Exceptional collections on arithmetic toric varieties

(joint with Matthew Ballard and Alexander Duncan)

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Tori

Let k be a field and $\bar{k}$ its algebraic closure.

Definition

A *k-torus* is an algebraic group T over k such that $T_{\bar{k}} := T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) \cong (\bar{k}^{\times})^n$.

Examples

- the split torus $\mathbb{G}_m^n = (k^{\times})^n$
- the circle group $S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}$; note that $S^1_{\mathbb{C}} \cong \mathbb{C}^{\times}$.
- the Weil restriction $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^{\times})$ (this is $\mathbb{C}^{\times}$ viewed as a real variety); note that $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^{\times})_{\mathbb{C}} \cong (\mathbb{C}^{\times})^2$.

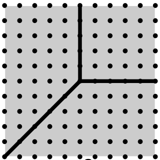
Arithmetic toric varieties

Definition

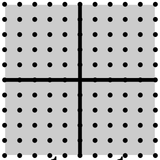
An arithmetic toric variety is a (smooth, projective) normal variety together with a faithful action of a torus which has a dense open orbit. A toric variety with torus T is split if T is a split torus.

Example: split toric varieties

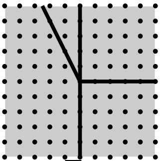
- (products of) projective space: $\mathbb{G}_m^n \subset \mathbb{A}^n \subset \mathbb{P}^n$
- if $T = \mathbb{G}_m^n$, then $X = X(\Sigma)$ for $\Sigma \subset \mathbb{Z}^n = \mathrm{Hom}(\mathbb{G}_m, T)$



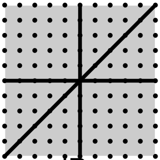
$\mathbb{P}^2$



$\mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{F}_a$



dP_6

Non-split arithmetic toric varieties

Example

- the real conic $\{(x:y:z) \in \mathbb{P}^2_{\mathbb{R}} \mid x^2+y^2+z^2=0\} = \mathrm{SB}(\mathbb{H})$
- toric variety with torus S^1
- over $\mathbb{C}$, this is $\mathbb{P}^1$ with torus $\mathbb{C}^\times$

Example

- real projective space $\mathbb{P}^1_{\mathbb{R}}$
- toric variety for two distinct tori: $\mathbb{R}^\times$ and S^1

Example

- the Weil restriction $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$
- toric variety with torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$
- over $\mathbb{C}$, this is $\mathbb{P}^1 \times \mathbb{P}^1$ with torus $(\mathbb{C}^\times)^2$

Exceptional collections

For a k-scheme X, let $\mathrm{D^b}(X) = \mathrm{D^b}(\mathrm{coh}(X))$ denote the bounded derived category of coherent sheaves on X.

Definition

An object E in $\mathrm{D^b}(X)$ is exceptional if

- $\mathrm{Ext}^n(E, E) = \mathrm{Hom}(E, E[n]) = 0$ for all $n \neq 0$, and
- $\mathrm{End}(E)$ is a division k-algebra of finite dimension

Definition

A sequence $\mathsf{E} = \{E_1, \ldots, E_s\}$ of exceptional objects is an exceptional collection if $\mathrm{Ext}^n(E_i, E_j) = 0$ for all n whenever $i > j$.

- E is full if it generates $\mathrm{D^b}(X)$.
- E is strong if $\mathrm{Ext}^n(E_i, E_j) = 0$ for all $n \neq 0$.

Exceptional collections

An exceptional collection $\{E_1, ..., E_n\}$ on a scheme X induces an isomorphism on K-theory

$$K_p(X) \cong \bigoplus_i K_p(D_i),$$

where $\text{End}(E_i) = D_i$.

Example

- (Beilinson) The set $\{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\}$ is a full strong exceptional collection of line bundles on $\mathbb{P}^n$.
- $\mathbb{P}^1 \times \mathbb{P}^1$ has full strong exceptional collection given by $\{\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)\}$, where $\mathcal{O}(i,j) = \pi_1^*\mathcal{O}(i) \otimes \pi_2^*\mathcal{O}(j)$.

Theorem (Kawamata)

Every split toric variety has a full exceptional collection.

Examples in non-split case

Example: the real conic $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$

- $X_{\mathbb{C}} \cong \mathbb{P}^1$
- $X = \mathrm{SB}(\mathbb{H})$, where $\mathbb{H}$ denotes Hamilton's quaternions.
- X has a full strong exceptional collection $\{\mathcal{O}_X, \mathcal{F}\}$
- $\mathrm{End}(\mathcal{F}) \cong \mathbb{H}$ and $\mathcal{F}_{\mathbb{C}} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Example: Weil restriction $Y = R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$

- $Y_{\mathbb{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$
- Y has a full strong exceptional collection $\{\mathcal{O}_Y, \mathcal{G}, \mathcal{H}\}$
- $\mathrm{End}(\mathcal{G}) \cong \mathbb{C}$ and $\mathcal{G}_{\mathbb{C}} \cong \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$
- $\mathrm{End}(\mathcal{H}) \cong \mathbb{R}$ and $\mathcal{H}_{\mathbb{C}} \cong \mathcal{O}(1,1)$

Descent of exceptional collections

Let X be a scheme with an action of a finite group G.

Definition

A set of objects E in $\mathrm{D}^{\mathrm{b}}(X)$ is G-stable if for all $A \in$ E and all $g \in G$ there exists $B \in$ E such that $g^*A \cong B$.

Theorem (Ballard, Duncan, M.)

Let X be a k-scheme and L/k a G-Galois extension. Then X_L admits a G-stable exceptional collection if and only if X admits an exceptional collection.

Moreover, if one collection is full/strong/of sheaves/of vector bundles, then so is the other.

Note that if E is a G-stable exceptional collection consisting of line bundles on X_L, the resulting collection on X may not consist of line bundles.

A question of Merkurjev and Panin

Definition

A G-lattice M is a free $\mathbb{Z}$-module with an action of G, i.e., a homomorphism $G \to \mathrm{GL}(M) \cong \mathrm{GL}_n(\mathbb{Z})$. A G-lattice is a permutation lattice if it has a basis which is permuted by G.

Theorem (Merkurjev, Panin '97)

Let X be a toric variety with splitting field L/k and $G = \mathrm{Gal}(L/k)$. Then $K_0(X_L)$ is a G-lattice which is a direct summand of a permutation lattice.

Question

Is $K_0(X_L)$ a permutation lattice?

Notice that the existence of a full exceptional collection on X provides an affirmative answer.

Main results

Theorem

The following possess full exceptional collections of sheaves:

- Severi-Brauer varieties (Bernardara '09)
- dP_6 (Blunk, Sierra, Smith '11)
- toric surfaces (Xie, BDM '17)
- toric Fano 3-folds (BDM)
- all forms of 43 of the 124 split toric Fano 4-folds (BDM)
- all forms of centrally symmetric toric Fano varieties (BDM; Castravet, Tevelev)
- all forms in characteristic zero of toric varieties corresponding to Weyl fans of root systems of type A (BDM; Castravet, Tevelev)

Remarks on methodology

X a toric variety, X_L split with fan Σ

- assume X_L admits an exceptional collection of line bundles
- $G = \mathrm{Gal}(L/k)$ acts on X_L so preserves Σ, and thus preserves $\Sigma(1)$
- $\Sigma(1) \leftrightarrow \mathrm{Div}_{T_L}(X_L)$
- this induces an action of G on $\mathrm{Pic}(X_L)$ which is completely determined by the action on Σ

Thus, to check G-stability, suffices to check $\mathrm{Aut}(\Sigma)$-stability.

- methods of Bondal and Uehara give stable collections for the 3-fold and 4-fold cases using the toric Frobenius
- the centrally symmetric results use an exceptional collection independently discovered by Castravet and Tevelev.
- the root system result builds on work of Castravet and Tevelev on equivariant derived categories.

¡Muchas gracias!