Yoga of Motives :

Given a class of invariants, construct a category which maps to all such invariants and which is universal w-r-t this property. This is taken as an anology from alg. topology.

Thus far, we have touched on 2 such classes

- Weil Cohomology theories. ⇝ ~-motives
- additive invariants ⇝ noncommutative motives.

Throughout let Sm/k denote the category of smooth schemes over k.

Voevodsky Motives

General idea: Instead of using topological phenomenon as analogy, transfer the study of cohomology ~~into~~ (and homotopy) of varieties into topological world and then use topological analysis directly.

At the same time, this theory needs to encode Chow theory (cycles and their intersections) so as to reflect the algebro-geometric structure.

That is, there should be 2 categories:

Sm/k = smooth schemes $/k$.

Motives

(stable) Homotopy Category

Voevodsky constructs

D (motives)

⊛ Contains Category of motives.

⊛ Cohomology theories of motives are representable

Motivic Cohomology.

The main ingredient here is the notion of "(pre)sheaves w/ transfers"

The main ingredient here is the notion of "simplicial sheaves"

We begin w/ an overview here.

Sheaves are defined by gluing properties relative a topology (a collection of open sets). Instead of using Zariski opens, we utilize a finer (but not too fine!) collection:

~~Nisnevich covering~~

A family of maps $\{p_i : U_i \to X\}$ is a <u>Nisnevich covering</u> of X if

⊛ p_i is étale (locally a covering space)

⊛ $\forall x \in X$, $\exists i$ and $u \in U$ so that $p_i(u) = x$ and the induced map $k(x) \longrightarrow k(u)$ is an isomorphism.

(arithmetic of cover isn't so different ~~relat~~ from the base).

Motivic Homotopy

Main idea: functor categories like

$$PSh(\mathcal{A}, \mathcal{B}) = \{ F : \mathcal{A}^{op} \to \mathcal{B} \}.$$

inherit many of the properties of the category $\mathcal{B}$.

* $PSh(\mathcal{A}, \underline{Sets})$ is (Co)complete.
* $Psh(\mathcal{A}, \underline{Ab})$ is abelian.

If we hope to do homotopy theory, we should consider sheaves valued in $\underline{Top}$ or $\underline{sSets}$.

The notion of a sheaf depends on what we call open sets (i.e. the topology). Let $\mathcal{T}$ be a given topology on Sm/k.

Consider $$Spc_{\mathcal{T}}(k) = \{ F : (Sm/k)^{op} \to \underline{sSets} \mid F \text{ is a } \mathcal{T}\text{-sheaf} \}.$$

and morphisms are morphisms of sheaves (i.e. natural transformations).

This category of "spaces" can also be defined as the category of simplicial objects in $Sh_\tau(Sm/_k, \underline{Sets})$, that is, the category of functors $\Delta^{op} \longrightarrow Sh_\tau(Sm/_k, \underline{Sets})$

where Δ = category of finite ordered set and order preserving functions.

For Voevodsky's theory, τ = Nisnevich topology.

<u>Q1</u>: How do we use this category to study schemes?

<u>A</u>: Yoneda!

$$Sm/_k \hookrightarrow Spc(k)$$
$$X \longmapsto h_X = Hom(-, X)$$

This is only a sheaf of sets but we may view it as a "const" simplicial sheaf

$$\left(Hom(-,X)\right)_n = Hom(-,X)$$

and the face and degeneracy maps are just the identity.

$f: X \to Y \;\rightsquigarrow\; h_X \to h_Y$ (via comp. w/ f)

which induces a map on the associated simplicial sets.

<u>Q2</u>: Can we use $Spc(k)$ to probe spaces topologically?

<u>A1</u>:

- <u>sSets</u> $\hookrightarrow Spc(k)$

$$S_* \longmapsto (U \longmapsto S_*)$$

The "constant" simplicial sheaf.

<u>A2</u>: Any construction/invariant in <u>sSet</u> (or <u>Top</u>) can be defined on $Spc(k)$ "pointwise".

Let $\mathcal{X} \in Spc(k)$. Define the <u>fundamental presheaf</u> as $U \longmapsto \pi_1(\mathcal{X}(U))$.

Then we can sheafify (rel Nisnevich or any τ).

This defines a sheaf of groups $\pi_1(\mathcal{X})$ on Sm/k.

Similarly, we can define

- higher homotopy groups
- homotopy classes of maps
- smash products, loop spaces, suspensions ...

by doing so pointwise and then sheafifying.

(the really interesting category is stable homotopy category and this is where we "invert $\mathbb{A}^1$," i.e. make it contractible)

Voevodsky Motives

Again borrowing from topology, instead of using subobjects to study and decompose varieties, we look to abelian sheaves which also reflect this subvariety structure (so called "sheaves w/ transfer").

From now on, let $Cor(X,Y)$ denote free abelian group generated by irreducible closed subsets whose associated integral subscheme is finite and surjective over X.

X, Y smooth schemes conn.

let $Cor(k)$ denote the associated category of correspondences

$$\underline{SmProj}(X) \longrightarrow Cor(k)$$

$$X \longmapsto X$$

$$f : X \to Y \qquad \Gamma_f \subseteq X \times Y.$$

Def$^{\underline{n}}$ A <u>presheaf w/ transfers</u>

is a functor

$$F : \operatorname{Cor}(k)^{op} \longrightarrow \underline{\underline{Ab}} .$$

Let $PST(k)$ denote the collection of all

such functors.

<u>Alternative Description</u> : Given $F \in PST(k)$

$$Sm/k^{op} \longrightarrow \operatorname{Cor}(k)^{op} \xrightarrow{F} \underline{Ab}$$

We can restrict to the category of smooth schemes.

This gives us a usual abelian presheaf

$$F : (Sm/k)^{op} \longrightarrow \underline{Ab}$$

$+$ extra maps coming from "generalized functions"

in $\operatorname{Cor}(k)$. That is, an extra "transfer" map

$F(Y) \to F(X)$ for each correspondence $X \rightsquigarrow Y$.

Examples:

⊛ Constant ~~sheaves~~ presheaves w/ transfers.

let $\underline{A} : Sm/_k^{op} \longrightarrow \underline{Ab}$ be the const. presheaf given by $X \longmapsto A$. Given a (prime) Corresp $X \rightsquigarrow Y$ ~~For any Correspondence~~ i.e. subset $Z \subseteq X \times Y$, we get homomorphisms

$$\begin{array}{ccc} \underline{A}(Y) & \longrightarrow & \underline{A}(X) \\ \| & & \| \\ A & \xrightarrow[\bullet \deg w/X]{} & A \end{array}$$

This defines a Const. presheaf w/ transfers.

⊛ Chow groups: $CH^i(-) : \underline{Cor}^{op} \longrightarrow \underline{Ab}$ define presheaves w/ transfers.

Of course, (from Keller or Candace's talks)

$$Z \subseteq X \times Y \rightsquigarrow CH^i(Y) \to CH^i(X)$$

via $W \longmapsto p_{1*}(Z \bullet p_2^* W)$.

⊛ Representable presheaves w/ transfers

$$\begin{array}{ccccc} Sm/k^{op} & \longrightarrow & Cor(k)^{op} & \longrightarrow & PST(k) \\ X & \longmapsto & X & \longmapsto & \mathbb{Z}_{tr}(X) \end{array}$$

defined by $\mathbb{Z}_{tr}(X)(U) = Cor(U, X)$

Given $f: V \to U$, we get $\mathbb{Z}_{tr}(X)(U) \to \mathbb{Z}_{tr}(X)(V)$

$$\begin{array}{ccc} \mathbb{Z}_{tr}(X)(U) & \to & \mathbb{Z}_{tr}(X)(V) \\ \| & & \| \\ Cor(U,X) & \longrightarrow & Cor(V,X) \end{array}$$

$\Gamma_f \subseteq V \times U$

$Z \subseteq U \times X \longmapsto p_{13*}\left(p_{12}^{*}\Gamma_f \cdot p_{23}^{*} Z\right).$

$$\begin{tikzcd} & V \times U \times X \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ \Gamma_f \subseteq V \times U & V \times X & U \times X \supseteq Z \end{tikzcd}$$

Summary: ~~[illegible]~~, presheaves w/ transfers allows to probe Sm/k w/ abelian sheaves which also reflect intersection-theoretic data. Now, we do homological algebra!

① Start w/ $Sh_{Nis}(Cor(k))$ = abelian Nisnevich sheaves w/ transfers.

② Consider the category $D^- = D^-(Sh_{Nis}(Cor(k)))$ of bounded above complexes of such sheaves.

Our category of motives should play nicely w/ our homotopy category, so we may invert products w/ $\mathbb{A}^1$

③ let $\mathcal{E}_{\mathbb{A}^1}$ = smallest thick (Serre) subcat of D^- containing all $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$ and closed under direct sums.

<u>Def$^{\underline{n}}$</u> The triangulated category of (effective) Motives over k is $DM^{eff,-}_{Nis}(k) = D^-(Sh_{Nis}(Cor(k)))/\mathcal{E}_{\mathbb{A}^1}$.

<u>Properties</u> : We let $M(X)$ denote class of $\mathbb{Z}_{tr}(X)$ in $DM(k)$.

⊛ (Mayer-Vietoris) $\{U, V\}$ cover of X smooth scheme

$\exists$ triangle in $DM(k)$

$$M(U \cap V) \to M(V) \oplus M(U) \to M(X) \to M(U \cap V)[1]$$

⊛ $E \to X$ vector bundle, $M(E) \to M(X)$.

⊛ $\exists$ projective bundle formula:

$\mathbb{P}(\mathcal{E}) \longrightarrow X$ proj. bundle of rank $(n+1)$.

$\exists$ isom $\displaystyle\bigoplus_{i=0}^{n} M(X)(i)[2i] \longrightarrow M(\mathbb{P}(\mathcal{E}))$.

⊛ (Blow up formula)

⊛ (Chow Motives) $Chow(k) \hookrightarrow DM(k)$.

Summary: We recover Chow motives and various formulae we expect from cycles but also get topological formulas. Furthermore, we get a (stable) homotopy category ~~to~~ to supplement our study of (derived) motives.