Compactifications of $\mathcal{M}_{o,n}$ from Rational Normal Curves

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Introduction-Moduli Problem

Recall that in our first construction of the compact space $\overline{\mathcal{M}}_{o,n}$, we took as our moduli problem the functor which associates to each scheme B a flat family of stable n-pointed trees of $\mathbb{P}^1$'s over B. Similarly, for the Hassett spaces $\overline{\mathcal{M}}_{0,\overrightarrow{c}}$, our moduli functor associates to each scheme B a flat family of stable $\overrightarrow{c}$-weighted trees of $\mathbb{P}^1$'s over B (with an appropriately modified stability condition). Notice that in each case our stability conditions only allow *nodal* singularities. That is, our singularities are locally analytically isomorphic to 2-dimensional coordinate axes.

We will now proceed in a similar fashion, with a slight modification to our allowable singularities. To each scheme B, we will associate a flat family of stable weighted trees of Veronese curves over B (also called *quasi-Veronese curves*). However, in this case our stability condition will allow for *multinodal* singularities, which are locally analytically isomorphic to n-dimensional coordinate axes.

1 The Veronese Embedding and Rational Normal Curves

Definition 1.1. Let $n, d \in \mathbb{Z}^+$. Let $M_0, ..., M_N$ be all the monomials of degree d in $(n+1)$ variables $x_0, ..., x_n$. Notice that $N = \binom{n+d}{d}$. For $a = [a_0 : \cdots : a_n]$, define

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

$$[a_o : \cdots : a_n] \mapsto [M_0(a) : \cdots : M_N(a)].$$

This gives an injective morphism of varieties called the *d-uple* or *Veronese embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. We can see that this map is well-defined, for if we take any other point in the equivalence class of $[a_0 : \cdots : a_n]$, say $[\lambda a_0 : \cdots : \lambda a_n]$ for $\lambda \in \mathbb{C}^*$, then we have

$$[\lambda a_0 : \cdots : \lambda a_n] \mapsto [\lambda^d M_0(a) : \cdots : \lambda^d M_N(a)],$$

which is equal to $[M_0(a) : \cdots : M_N(a)]$ in $\mathbb{P}^N$. It is a morphism as it is a polynomial map in each coordinate. Injectivity is left as an exercise.

In the case $n = 1$, we have

$$\nu_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$
$$[a_0 : a_1] \mapsto [a_0^d : a_0^{d-1}a_1 : a_0^{d-2}a_1^2 : \cdots : a_0a_1^{d-1} : a_1^d].$$

The image of ν_d is called *the rational normal curve of degree d in $\mathbb{P}^d$.*

Example 1.2. (Twisted Cubic) Consider the 3-uple embedding

$$\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

given by

$$[a_0 : a_1] \mapsto [a_0^3 : a_0^2a_1 : a_0a_1^2 : a_1^3].$$

We would like to see that $\operatorname{Im}\nu_3 \subset \mathbb{P}^3$ is in fact a projective subvariety of $\mathbb{P}^3$. That is, $\operatorname{Im}\nu_3 = Z(\mathfrak{q})$ for some homogeneous ideal $\mathfrak{q} \lhd k[x, y, z, t]$. Let

$$\mathfrak{q} = (xt - yz, y^2 - xz, z^2 - yt).$$

Of course $\mathfrak{q}$ is homogeneous, as it is generated by homogeneous elements. To see that $\operatorname{Im}\nu_3 = Z(\mathfrak{q})$, we notice that one inclusion is clear: if $[b_0 : b_1 : b_2 : b_3] \in \operatorname{Im}\nu_3$, then

$$b_0 = a_0^3,$$
$$b_1 = a_0^2a_1,$$
$$b_2 = a_0a_1^2,$$
$$b_3 = a_1^3$$

for some $[a_0 : a_1] \in \mathbb{P}^1$. Then of course $[b_0 : b_1 : b_2 : b_3]$ satisfies the defining equations of $\mathfrak{q}$ and so $\operatorname{Im}\nu_3 \subset Z(\mathfrak{q})$. The reverse inclusion is left as an exercise to the reader.

For a visual interpretation, let us consider the affine patch where $a_0 = 1$. Then our twisted cubic will have points of the form

$$(a_0^3, a_0^2a_1, a_0a_1^2, a_1^3) = (1, a_1, a_1^2, a_1^3).$$

Thus, the graph of our twisted cubic is given by

$$\{(a, a^2, a^3) \in \mathbb{A}^3 \mid a \in \mathbb{C}\}.$$

2 $U_{d,n}$ and the $\mathbf{SL}_{d+1}$-Action

Recall that the Hilbert scheme $\operatorname{Hilb}^{p(x)}(X)$ is a scheme whose points correspond to closed subschemes of X with Hilbert polynomial $p(x)$. We will be concerned with the compact Hilbert scheme $\operatorname{Hilb}^{dx+1}(\mathbb{P}^d)$. For if $X \in \operatorname{Hilb}^{dx+1}(\mathbb{P}^d)$, we have seen that for $n \in \mathbb{Z}$,

$$p(n) = \chi(\mathcal{O}_X(n)).$$

Thus, we have

$$\begin{aligned} 1 &= p(0) \\ &= \chi(\mathcal{O}_X) \\ &= h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) \\ &= 1 - h^1(\mathcal{O}_X) \\ &= 1 - g \end{aligned}$$

and so X is genus 0. Since $\deg(dx+1) = 1$ we have $\dim X = 1$. Also,

$$\begin{aligned} \deg X &= [\deg p(x)]! \cdot [\text{leading coefficient of } p(x)] \\ &= 1! \cdot d \\ &= d \end{aligned}$$

Thus, $\text{Hilb}^{dx+1}(\mathbb{P}^d)$ contains all genus 0 curves of degree d in $\mathbb{P}^d$ and so contains V_d, the collection of degree d rational normal curves. Let $\mathscr{H}_d \subset \text{Hilb}^{dx+1}(\mathbb{P}^d)$ denote the closed component of rational normal curves of degree d and their degenerations. One may identify $\mathscr{H}_d$ as the closure of V_d in $\text{Hilb}^{dx+1}(\mathbb{P}^d)$.

We may also describe this locus in terms of another object: the Chow variety. Define the Chow variety, $\text{Chow}(1, d, \mathbb{P}^d)$, as the variety which parameterizes curves or 1-cycles ($\mathbb{Z}$-linear combinations of 1-dimensional, irreducible, closed subschemes) of degree d in $\mathbb{P}^d$. We will use $\mathscr{C}_d$ to denote the irreducible component which parameterizes rational normal curves and their limit cycles or degenerations. Define

$$U_{d,n} = \{(X, p_1, ..., p_n) \in \mathscr{C}_d \times (\mathbb{P}^d)^n \mid p_i \in X \ \forall i\}.$$

That is, an element of $U_{d,n}$ is a quasi-Veronese curve or tree of rational normal curves which has n marked points.

Example 2.1. Let X be the twisted cubic above and let $p = [1 : 1 : 1 : 1]$. Then $(X, p) \in U_{3,1}$. If $q = [\frac{1}{2} : 1 : 2 : 4]$, then $(X, p, q) \in U_{3,2}$.

There is a natural action of SL_{d+1} on $U_{d,n}$. To see this, let us think of $\mathbb{P}^d$ as the projectivization of a $(d+1)$-dimensional vector space V. That is, $\mathbb{P}^d \cong \mathbb{P}V$, where

$$\mathbb{P}V := \{W \leq V \mid \dim W = 1\}.$$

We have an action of GL_{d+1} on V by left matrix multiplication. Define the projective linear group

$$\text{PGL}_{d+1} := \text{GL}_{d+1} / Z(\text{GL}_{d+1}),$$

where $Z(\text{GL}_{d+1})$ is the center of GL_{d+1}. Then we have an induced action of PGL_{d+1} on $\mathbb{P}V$. In particular, the only difference between the PGL_{d+1}-action and the GL_{d+1}-action is that the former is *effective*. That is, no nontrivial element of PGL_{d+1} acts trivially. Another way of saying this is that if we act by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

then multiplying each entry of our matrix by a nonzero scalar α gives

$$\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha A \\ \alpha B \end{pmatrix},$$

which is the same point as (A, B) in projective space. Since these matrices act the same, we identify them in the quotient. Finally, we have an inclusion

$$\mathrm{SL}_{d+1} \hookrightarrow \mathrm{GL}_{d+1}.$$

Following this with the quotient map

$$\mathrm{SL}_{d+1} \hookrightarrow \mathrm{GL}_{d+1} \twoheadrightarrow \mathrm{PGL}_{d+1}$$

yields an action of SL_{d+1} on $\mathbb{P}^d$. The reason we would like to have an action of SL_{d+1} is that many theorems concerning quotients by group actions (GIT quotients) require that the acting group be semisimple.

3 GIT Quotients

We now want to introduce the quotient spaces $U_{d,n}/\!/\,\mathrm{SL}_{d+1}$. We will see that these are alternative compactifications of $M_{0,n}$ and solve the moduli problem given above. Moreover, these spaces receive maps from the Hassett spaces discussed earlier. Unfortunately, defining the quotient for schemes and varieties by a group action tends to be a complicated matter. In some cases, the quotient does not properly detect the different orbits given by the group action. We give an example of this situation below. In general, given a variety and a group action, the quotient by this group action may not actually be a variety. To fix this problem, we take the quotient of a subset of our variety which consists of "good" points. That is, we throw out "bad" points and only consider points which are well-behaved under the group action.

Example 3.1. Let $\mathbb{C}^*$ act on $\mathbb{A}^2$ via $\lambda \cdot (a_1, a_2) = (\lambda a_1, \lambda a_2)$. Then the orbits of this action are lines through the origin. In particular, all of the orbits intersect, and thus our quotient is a single point.

However, if we let $\mathbb{C}^*$ act on $\mathbb{A}^2 \setminus O$, our quotient becomes $\mathbb{P}^1$. In this case, the smallest closed, $\mathbb{C}^*$-invariant subsets are exactly the orbits of our action. Additionally, two different orbits are mapped to two different points, contrary to the result of the action on $\mathbb{A}^2$ above. A quotient with such desirable properties is called a *good geometric quotient*. In this situation, we see that O is a bad point with respect to this action. In the talk to follow, we examine how to distinguish good points and bad points and explore the machinery of GIT (geometric invariant theory) quotients.

References

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