Notes from the UGA K-Theory Seminar

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Introduction

*To be completed.

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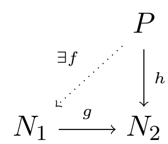
1 K_0 of a Ring (J. Tenini)

1.1 General Construction

We will assume basic familiarity with groups, rings and modules.

Definition 1.1. Let R be a ring. An R-module P is said to be *projective* if it satisfies the following equivalent conditions:
(1). There is an R-module Q such that $P \oplus Q$ is free.
(2). Given modules N_1, N_2 and R-module homomorphisms $g : N_1 \longrightarrow N_2$ surjective and $h : P \longrightarrow N_2$, there exists a unique R-module homomorphism $f : P \longrightarrow N_1$ such that the following diagram commutes:



(3). Given an R-module N and a surjective R-module homomorphism $\pi : N \longrightarrow P$, then there is an injective R-module homomorphism $i : P \longrightarrow N$ such that $\pi \circ i = \mathrm{Id}_P$.

Remark 1.2. In particular, we will only be interested in *finitely generated* projective modules.

Example 1.3. The most obvious class of projective modules are free modules. Free modules are clearly projective if we consider Definition 1.1 (1), taking Q to be the trivial module.

Example 1.4. If R is a principal ideal domain and M is a finitely generated R-module, a basic structure theorem from linear algebra yields the isomorphism $M \cong F \oplus M_t$ where F is a free R-module and M_t is a torsion R-module. As stated above, a free module is projective. When R is a PID, we have that the converse is also true. For if P is a projective R-module, there is some R-module Q with $P \oplus Q$ free. Then M is torsion-free and by the aforementioned structure theorem we have that M is free.

At this point, the reader may be wondering if in fact all projective modules are free. The following example shows this is not the case.

Example 1.5. $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}/6\mathbb{Z}$-module is projective but not free.

Definition 1.6. Let R be a ring. We define the *Grothendieck Group* K_0R as the Abelian group with the following generators and relations:
(G): Take one generator $[P]$ for each isomorphism class in the category of finitely generated projective R-modules.
(R): $[P] + [Q] = [P \oplus Q]$ for every pair, P, Q, of finitely generated projective R-modules.

Remark 1.7. Each element of K_0R may be written in the form $[P] - [Q]$.

Proposition 1.8. *If R is a PID, then $K_0R \cong \mathbb{Z}$.*

Proof. Over a PID, an R-module is projective if and only if it is free. Thus, projective R-modules are determined up to isomorphisms by their rank, and the direct sum of projective modules corresponds to the sum of the respective ranks. □

Definition 1.9. Let P and Q be finitely generated projective R-modules. We say P and Q are *stably isomorphic* if there is some $n \in \mathbb{Z}^+$ such that $P \oplus R^n \cong Q \oplus R^n$.

Proposition 1.10. *Let P and Q be finitely generated projective R-modules. Then $[P] = [Q]$ in K_0R if and only if P and Q are stably isomorphic.*

Proof. ($\Leftarrow$) If $P \oplus R^n \cong Q \oplus R^n$, then $[P] + [R^n] = [Q] + [R^n]$ in K_0R. Since K_0R is a group, it follows that $[P] = [Q]$.
($\Rightarrow$) *To be completed. □

1.2 The Tensor Product

When R is commutative, we have an equivalence of categories between $\mathbf{Mod_R}$, the category of right R-modules, and $_{\mathbf{R}}\mathbf{Mod}$ the category of left R-modules. In this situation, the tensor product makes K_0R into a ring. Also, given any homomorphism of rings $f : R \longrightarrow S$, the tensor product enables one to view R-modules as S-modules in a natural way, giving rise to the induced homomorphism of groups $K_0f : K_0R \longrightarrow K_0S$.

Definition 1.11. Let R, S, and T be rings. Let M be an R-S bimodule and N an S-T bimodule. Define $M \otimes_S N$ to be the following R-T bimodule:
As an Abelian group, it is generated by the symbols $m \otimes n$ ($m \in M$, $n \in N$) with the following relations:
(1). $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$
(2). $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$
(3). $ms \otimes n = m \otimes sn$.

The left action is given by $r(m \otimes n) = rm \otimes n$ and the right action is given by $(m \otimes n)t = m \otimes nt$.

Remark 1.12. Alternatively, if one prefers the language of category theory, one can define the tensor product $M \otimes_S N$ as the initial object in the category $\mathcal{C}$ where

$$\text{Ob}(\mathcal{C}) = \{\varphi : M \times N \longrightarrow A \mid \varphi \,\text{is balanced and bilinear and}\, A \,\text{is an}\, R - T \,\text{bimodule}\}$$

and

$$\text{Mor}(\mathcal{C}) = \{\psi : F \longrightarrow G \mid F \,\text{and}\, G \,\text{are}\, R - T \,\text{bimodules}\}.$$

Proposition 1.13. *Let R be a commutative ring and let P and Q be finitely generated projective R-modules. Then P, Q and $P \otimes_R Q$ are R-R bimodules. Moreover, $P \otimes_R Q$ is a finitely generated projective R-module.*

Proof. The R-R bimodule structure of P, Q and $P \otimes_R Q$ is clear. Let P' and Q' be R-modules such that $P' \oplus P \cong R^n$ and $Q' \oplus Q \cong R^m$. Then

$$\begin{aligned}(P \otimes_R Q) \oplus (P \otimes_R Q') \oplus (P' \otimes_R Q) \oplus (P' \otimes_R Q') &\cong (P \oplus P') \otimes_R (Q \oplus Q') \\ &\cong R^n \otimes_R R^m \\ &\cong R^{nm}.\end{aligned}$$

Thus, $P \otimes_R Q$ is free. □

Remark 1.14. One can check that defining $[P][Q] = [P \otimes_R Q]$ makes K_0R into a ring.

Let $f : R \longrightarrow S$ be a homomorphism of rings. One can then think of S as a right R-module via $s \cdot r = sf(r)$. One can also regard S as a left S-module and so S is an S-R bimodule. Thus, if P is an R-module, define $S \otimes_f P := S \otimes_R P$.

Remark 1.15. $S \otimes_f R \cong S$

Remark 1.16. If P is finitely generated projective R-module, and $P \oplus Q \cong R^n$, then

$$\begin{aligned}(S \otimes_f P) \oplus (S \otimes_f Q) &\cong S \otimes_f (P \oplus Q) \\ &\cong S \otimes_f R^n \\ &\cong (S \otimes_f R)^n \\ &\cong S^n.\end{aligned}$$

Thus, $S \otimes_f P$ is a finitely generated projective S-module.

We can now define the induced group homomorphism $K_0 f : K_0R \to K_0S$, via $[P] \mapsto [S \otimes_f P]$. Thus, one can define K_0 as a functor $K_0 : \mathbf{Ring} \longrightarrow \mathbf{Ab}$.

2 K_0 as a Functor (J. Tenini)

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a mapping which to each object $X \in \mathrm{Ob}(\mathcal{C})$ associates an object $F(X) \in \mathrm{Ob}(\mathcal{D})$, and to each morphism $f : X \longrightarrow Y$ associates a morphism $F(f) : F(\mathcal{C}) \longrightarrow F(\mathcal{D})$ satisfying the following:

(FUN 1). $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$, for all $X \in \mathrm{Ob}(C)$,

(FUN 2). $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$.

Example 2.2. Some covariant functors: Fundamental Group, $\pi_1 : \mathbf{Top} \longrightarrow \mathbf{Grp}$; Homology (simplicial, singular, cellular), $H_n(\bullet, R) : \mathbf{Top} \longrightarrow \mathbf{Mod_R}$.

Definition 2.3. A *contravariant functor* $G : \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor $G : \mathcal{C} \longrightarrow \mathcal{D}^{\mathrm{op}}$.

Example 2.4. Some contravariant fuctors: Global Sections, $\Gamma : \mathbf{Var} \longrightarrow \mathbf{Rings}$; Spectrum of a ring, $\mathrm{Spec} : \mathbf{CRings} \longrightarrow \mathbf{Top}$.

In the previous lecture, it was shown that if P is a finitely generated projective R-module and if $f : R \longrightarrow S$ is a homomorphism of rings, then $S \otimes_f P$ is a finitely generated projective S-module. It remains to show that this map $([P] \mapsto [S \otimes_f P])$ is well-defined.

Let $[P] = [Q]$. Then $R^n \oplus P \cong R^n \oplus Q$ for some $n \in \mathbb{Z}$. Then we have

$$S \otimes_f (R^n \oplus P) \cong S \otimes_f (R^n \oplus Q),$$

$$\Rightarrow (S \otimes_f R^n) \oplus (S \otimes_f P) \cong (S \otimes_f R^n) \oplus (S \otimes_f Q),$$

$$\Rightarrow S^n \oplus (S \otimes_f P) \cong S^n \oplus (S \otimes_f Q).$$

Thus, our map is well-defined.

We also need to check that the morphism $K_0 f$ induced by the ring homomorphism $f : R \longrightarrow S$ is in fact a group homomorphism.

Let $[P]$ and $[Q]$ be in $K_0 R$. Then we have

$$\begin{aligned} K_0 f([P] + [Q]) &= K_0([P \oplus Q]) \\ &= [S \otimes_f (P \oplus Q)] \\ &= [(S \otimes_f P) \oplus (S \otimes_f Q)] \\ &= [(S \otimes_f P)] + [(S \otimes_f Q)] \\ &= K_0 f([P]) + K_0 f([Q]). \end{aligned}$$

Remark 2.5. If R and S are commutative rings then $K_0 f$ is a homomorphism of rings.

Finally, we need to ensure that K_0 satsifies the properties of functors stated in Definition 2.1 above.

(FUN 1) Let $\mathrm{Id}_R : R \longrightarrow R$ be the identity map.

$$\begin{aligned} K_0(\mathrm{Id_R})([P]) &= [R \otimes_{\mathrm{Id}_R} P] \\ &= [R \otimes_R P] \\ &= [P]. \end{aligned}$$

(FUN 2) Let $f : R \longrightarrow S$ and $g : S \longrightarrow T$ be ring homomorphisms. Then

$$\begin{aligned} K_0(g \circ f)([P]) &= [T \otimes_{g \circ f} P] \\ &= [(T \otimes_g S) \otimes_f P] \\ &= [T \otimes_g (S \otimes_f P)] \\ &= (K_0(g) \circ K_0(f))([P]), \end{aligned}$$

as desired.
Thus, $K_0 : \mathbf{Rings} \longrightarrow \mathbf{Ab}$ and $K_0 : \mathbf{CRings} \longrightarrow \mathbf{CRings}$ are functors.

3 K_0 of an Abelian Category (J. Tenini)

Definition 3.1. An additive category $\mathcal{C}$ is a category satisfying the following conditions:
(1). $\mathcal{C}$ contains a 0 object, i.e. an object that is both initial and final.
(2). $\mathcal{C}$ contains all finite products $A \times B$
(3). Every set $\mathrm{Hom}(A, B)$ is given the structure of an abelian group in such a way that composition is bilinear.

Remark 3.2. It follows in fact that in an additive category, finite products and coproducts exist and are the same.

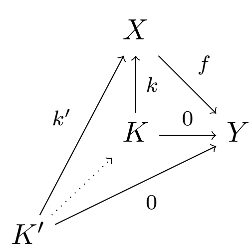
Remark 3.3. Composition being bilinear means that for maps, $f, f' : A \longrightarrow B$, $g, g' : B \longrightarrow C$, then

$$(g + g') \circ (f + f') = g \circ f + g' \circ f + g \circ f' + g' \circ f'$$

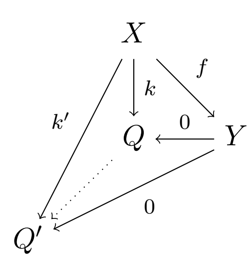
Example 3.4. The archetypal example to bear in mind throughout this discussion of additive categories is, of course, the category of Abelian groups.

Definition 3.5. An additive Abelian category $\mathcal{A}$ is an additive category in which:
(1). Every morphism $f : B \longrightarrow C$ has a kernel and cokernel.
(2). Every monomorphism is a kernel and every epimorphism is a cokernel.

Definition 3.6. The kernel of a morphism $f : X \longrightarrow Y$ is an object K with a morphism $k : K \longrightarrow X$ such that for any other $k' : K' \longrightarrow X$, we have a unique map $g : K' \longrightarrow K$ such that the following diagram commutes:



Definition 3.7. The cokernel of a morphism $f : X \longrightarrow Y$ is an object Q with a morphism $q : Y \longrightarrow Q$ such that for any other $q' : Y \longrightarrow Q'$, we have a unique map $g : Q \longrightarrow Q'$ such that the following diagram commutes:



Definition 3.8. A morphism f is a monomorphism if:

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

Definition 3.9. A morphism f is an epimorphism if:

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

Remark 3.10. It is a useful exercise to verify that these category theoretic terms have the following translations when working with Abelian groups:
(1). Monomorphism = Injective Homomorphism
(2). Epimorphism = Surjective Homomorphism
(3). Kernel = Kernel
(4). Cokernel = Codomain/Image

Definition 3.11. In an Abelian category, we say a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if $\ker g = \operatorname{im} f := \ker(B \longrightarrow \operatorname{coker} f)$

Definition 3.12. A category C is *small* if both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets (and not proper classes). A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a skeleton of $\mathcal{C}$ if the inclusion functor is an equivalence of categories and no two objects of $\mathcal{S}$ are isomorphic. A category $\mathcal{C}$ is said to be *skeletally small* if there is a skeleton $\mathcal{S}$ of $\mathcal{C}$ that is small.

Definition 3.13. Let $\mathcal{A}$ be a skeletally small additive Abelian category. Its Grothendieck Group $K_0(\mathcal{A})$ is the Abelian group presented as having one generator $[A]$ for each object, with one relation

$$[A] = [A'] + [A'']$$

for every short exact sequence:

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

Example 3.14. Let's compute $K_0(\mathcal{A})$ where $\mathcal{A}$ is the category of finitely generated Abelian groups: By the structure theorem for finitely generated modules over a PID, if G is a finitely generated Abelian group, then we have:

$$G = \mathbb{Z}^n \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_t\mathbb{Z}$$

Moreover, for any $k \in \mathbb{Z}^+$, the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow 0$$

gives us that $[\mathbb{Z}] = [\mathbb{Z}] + [\mathbb{Z}/k\mathbb{Z}]$ and so $[\mathbb{Z}/k\mathbb{Z}] = [0]$. Thus, using the structure theorem, we have that $K_0(\mathcal{A}) \cong \mathbb{Z}$.

4 Vector Bundles I (A. Brunyate)

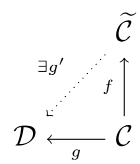
Throughout this section, let k be a field.

Note that the category of free R-modules form a full subcategory of the category of R-modules. Our goal is to define K_0 in this subcategory.

Definition 4.1. A *pseudo-Abelian category* $\mathcal{C}$ is an additive subcategory such that every idempotent splits.

That is, if $p : E \longrightarrow E$ is a morphism in $\mathcal{C}$ such that $p^2 = p$ then $\ker(p)$ and $\ker(1-p)$ exist and $E = \ker(p) \oplus \ker(1-p)$.

Theorem 4.2. *If $\mathcal{C}$ is an additive category, there exists a pseudo-Abelian category $\widetilde{\mathcal{C}}$ and an additive functor $f : \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ such that if $\mathcal{D}$ is any other pseudo-Abelian category and $g : \mathcal{C} \longrightarrow \mathcal{D}$ is additive, then there exists a functor $g' : \widetilde{\mathcal{C}} \longrightarrow \mathcal{D}$ making the following diagram commute:*



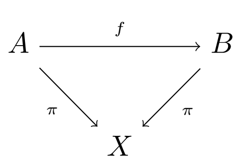
Theorem 4.3. *If $\mathcal{C}$ is an additive category, $\mathcal{D}$ a pseudo-Abelian category and $g : \mathcal{C} \longrightarrow \mathcal{D}$ a fully faithful additive functor where every object of $\mathcal{D}$ is a summand of an object in the image of g, then the map g', defined above, is an equivalence of categories.*

Example 4.4. Let $\mathcal{C}$ be the category of finitely generated *free* R-modules. Then $\widetilde{\mathcal{C}}$ is the category of finitely generated *projective* R-modules.

Example 4.5. Let X be a compact topological space. Let $\mathcal{D}$ be the category of trivial vector bundles over X. Then $\widetilde{\mathcal{D}}$ is the category of vector bundles over X.

Definition 4.6. A *quasi-vector bundle with base space X and fibre k^n* is a topology on $\coprod E_x$ such that the natural projection $\pi : E \longrightarrow X$ is continuous.

Definition 4.7. Let A and B be vector bundles over a topological space X. A *morphism of quasi-vector bundles with base space* X is a map $f : A \longrightarrow B$ such that $f_x : A_x \longrightarrow B_x$ is k-linear and such that the following diagram commutes:



Definition 4.8. A *trivial* quasi-vector bundle over X is the space $X \times k^n$ with the obvious projection.

4.1 Maps Between Trivial Vector Bundles

Theorem 4.9. *Let V_1, V_2 be finite-dimensional vector spaces over k. A map $\hat{g} : X \longrightarrow \mathrm{Hom}(V_1, V_2)$ corresponds to the map g_x on fibres for some quasi-vector bundle morphism $g : X \times V_1 \longrightarrow X \times V_2$ if and only if $\hat{g}$ is continuous.*

Definition 4.10. A *vector bundle* A over X is a quasi-vector bundle over X such that there exists an open cover $\{U_i\}_{i \in I}$ such that $A|_{U_i}$ is isomorphic to a trivial bundle.

Example 4.11. The Möbius band is a vector bundle over S^1.

5 Vector Bundles II (A. Brunyate)

Example 5.1. (The Canonical Bundle on $\mathbb{P}^{n-1}$) This is defined as the following subset $\mathbb{P}^{n-1} \times k^n$ where the point (x, y) is in the subset if and only if y is on the line corresponding to x.

5.1 Pullback of Vector Bundles

If $f : X \longrightarrow Y$ is a map of topological spaces and E is a vector bundle over Y, we define

$$f^*(E) := X_f \times_\pi E$$

i.e. $f^*(E) = \{(x, e) \in X \times E \mid f(x) = \pi(e)\}$.

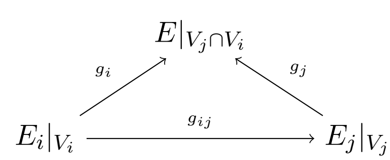
Example 5.2. If $X \longrightarrow \mathbb{P}^n$, we can pull back the canonical bundle on $\mathbb{P}^n$ to get a special line bundle on X.

5.2 Clutching of Bundles

Definition 5.3. Let $\{V_i\}_{i \in I}$ be an open cover of X and E_i a vector bundle over V_i with projection map π_i for all $i \in I$. Given isomorphism $g_{ij} : E_i|_{V_i \cap V_j} \longrightarrow E_j|_{V_i \cap V_j}$ satisfying

$$g_{ki}|_{V_i \cap V_j \cap V_k} = g_{kj}|_{V_i \cap V_j \cap V_k} \circ g_{ji}|_{V_i \cap V_j \cap V_k}$$

Then there exists a vector bundle E over X and isomorphisms $g_i : E_i \longrightarrow E|_{V_j}$ such that the following diagram commutes:



E here is called the clutching of the bundles E_i.

Example 5.4. (Tangent Bundles to differentiable manifolds:) Let U_i be an atlas for M, the tangent bundle is obtained by gluing $U_i \times \mathbb{R}^n$ using the derivative of the transition maps on fibers.

Definition 5.5. Let $\mathcal{E}$ be the category of finite dimensional k-vector spaces. We say that a functor $\mathcal{E} \longrightarrow \mathcal{E}$ is continuous if it induces a continuous map on Hom-spaces.

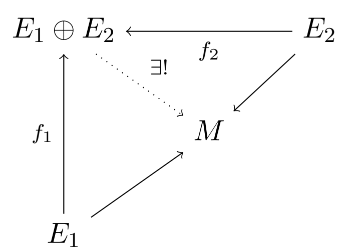
Example 5.6. Tensor Product, Sum, etc.

So, given a continuous functor $\phi : \mathcal{E} \longrightarrow \mathcal{E}$, we can define a functor $\phi' : \mathcal{E}(X) \longrightarrow \mathcal{E}(X)$ on the category of vector bundles over X by taking a trivialization cover of a vector bundle E over X and building $\phi'(E)$ by gluing together $U_i \times \phi(k^n)$ using ϕ applied to the fiber maps in the original gluing.

Example 5.7. The functor $E \times E \longrightarrow E$ defined by $(V_1, V_2) \longrightarrow V_1 \oplus V_2$ gives a functor

$$E(X) \times E(X) \longrightarrow E(X)$$

which we denote by $\oplus$.



Theorem 5.8. *Let E be a vector bundle over X, $p : E \longrightarrow E$ a map of vector bundles such that $p^2 = p$. Then $\ker(p)$ is a vector bundle over X.*

Proof. We need to show local triviality. Since this is a local property, we can assume that $E = X \times k^n$.
Trick: Write $f(x) = 1 - p_x - p_{x_0} + 2p_x p_{x_0}$ for some point $x_0 \in X$.
Then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker p & \longrightarrow & X \times k^n & \xrightarrow{p} & X \times k^n \\ & & & & \downarrow \tilde{f} & & \downarrow \tilde{f} \\ 0 & \longrightarrow & X \times \ker p_x & \longrightarrow & X \times k^n & \longrightarrow & X \times k^n \end{array}$$

□

6 Algebraic Vector Bundles (P. McFaddin)

Throughout this section R will denote a commutative ring with identity and k will denote an algebraically closed field.

We will continue to emphasize the breadth of K-theory by applying the theory to schemes and varieties. After defining algebraic vector bundles on schemes (and therefore on varieties), we will arrive at the scheme-theoretic analogue of the following theorem of Serre:

Theorem 6.1. *(Serre, 1955) Let V be an affine algebraic variety. Then there is a one-to-one correspondence between vector bundles over V and finitely generated projective modules over $\Gamma(V) = k[x_1, ..., x_n]/I(V)$, the coordinate ring of V.*

We begin with a slew of definitions.

Definition 6.2. Let X be a topological space. A *presheaf*, $\mathscr{F}$, of rings on X consists of the following data:
(a). For each open $U \subseteq X$, a ring $\mathscr{F}(U)$.
(b). For each inclusion of open sets $V \subseteq U$, a ring homomorphism $\rho_{UV} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$.

subject to the conditions

(0). $\mathscr{F}(\emptyset) = (0)$, the zero ring.
(1). $\rho_{UU} = \mathrm{Id}_{\mathscr{F}(\mathrm{U})}$.
(2). If $W \subseteq V \subseteq U$ then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Definition 6.3. A *sheaf* is a presheaf which also satisfies the following condition:
For each open set $U \subseteq X$, if $\{U_i\}_{i \in I}$ is an open cover of U, and if we have $s_i \in \mathscr{F}(U_i)$ for each i, with the property that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each i, $j \in I$, then there is a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

Example 6.4. Let X be a topological space. Let $\mathcal{O}_{\mathrm{top}}(U) = \{f : U \longrightarrow \mathbb{R} \mid f \text{ continuous}\}$. Then $\mathcal{O}_{\mathrm{top}}$ is a sheaf on X. Similarly, we may define the sheaf of differentiable functions on a differentiable manifold or the sheaf of holomorphic functions on a complex manifold.

Definition 6.5. A *morphism of presheaves* $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ is a homomorphism of rings $\varphi(U) : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ for each open set U such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho_{UV} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, we use the same definition for a morphism of sheaves.

Definition 6.6. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. For any sheaf $\mathscr{F}$ on X we define the *direct image sheaf* $f_*\mathscr{F}$ on Y by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.

Definition 6.7. A *ringed space* $(X, \mathcal{O}_X)$ is topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ consisting of a continuous map $f : X \longrightarrow Y$ and a morphism of sheaves $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.

Definition 6.8. A *locally ringed space* is a ringed space such that for each point $P \in X$, the stalk $\mathcal{O}_{X,P} := \varinjlim_{p \in U} \mathcal{O}_X(U)$ is a local ring (has a unique maximal ideal $\mathfrak{m}_P$). A *morphism of locally ringed spaces* is a morphism of ringed spaces which also satisfies the condition that $f_P^{\#} : \mathcal{O}_{Y,f(P)} \longrightarrow (f_*\mathcal{O}_X)_P$ is a local homomorphism of rings. That is, $(f_P^{\#})^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$

Example 6.9. Let R be a ring. Let $\operatorname{Spec} R$ denote the set of prime ideals of R. Let $\mathfrak{a}$ be an ideal in R. Let $V(\mathfrak{a})$ denote the set of all prime ideals containing $\mathfrak{a}$. We may define a topology on $\operatorname{Spec} R$ by taking sets of the form $V(\mathfrak{a})$ to be the closed subsets of $\operatorname{Spec} R$. Now, define a sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} R$ by setting

$$\mathcal{O}(U) = \{s : U \longrightarrow \bigcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}\}$$

such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ for each $\mathfrak{p}$ and s is locally a quotient of elements of R (here, $R_{\mathfrak{p}}$ is the localization of R at $\mathfrak{p}$). Then $(\operatorname{Spec} R, \mathcal{O})$ is a locally ringed space.

Definition 6.10. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $\operatorname{Spec} R$ for some ring R.

Definition 6.11. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $P \in X$ there is an open set U containing P such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. A morphism of schemes is a morphism of locally ringed spaces.

Example 6.12. $\operatorname{Spec} k[x_1, ..., x_n] =: \mathbb{A}^n_k$ is an affine scheme and $\operatorname{Spec} \operatorname{Proj} k[x_0, ..., x_n] =: \mathbb{P}^n_k$ is a non-affine scheme. For a definition and construction of the latter see [Hart].

Definition 6.13. A *sheaf of $\mathcal{O}_X$-modules* or an $\mathcal{O}_X$-module $\mathscr{F}$ is a sheaf on X which satisfies the following conditions:
(1). For each open $U \subset X$, $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module.
(2). For each inclusion $U \subseteq V$, the restriction map $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is an $\mathcal{O}_X(U)$-module homomorphism.
A *morphism of $\mathcal{O}_X$-modules* $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ is a sheaf morphism such that $\mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is $\mathcal{O}_X(U)$-linear.

Construction 6.14. Let R be a ring and let M be an R-module. We wish to construct the *sheaf associated to* M on $\operatorname{Spec} R$, denoted $\widetilde{M}$. This construction is quite similar to that of the sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} R$ given above. We proceed as follows: Let $\mathfrak{p} \in \operatorname{Spec} R$ and let $M_{\mathfrak{p}}$ be the localization of M at $\mathfrak{p}$. For $U \subseteq \operatorname{Spec} R$, let $\widetilde{M}(U) = \{s : U \longrightarrow \bigcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}\}$ such that $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ for each $\mathfrak{p}$ and for each $\mathfrak{p} \in U$ there exists a neighborhood V of $\mathfrak{p}$, there exists $m \in M$ and $f \in R$ such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f}$ in $M_{\mathfrak{q}}$.

Example 6.15. $\mathcal{O}_X$ is an $\mathcal{O}_X$-module. $\widetilde{M}$ defined above is an $\mathcal{O}$-module, where $\mathcal{O}$ denotes the sheaf of rings defined on $\operatorname{Spec} R$.

Example 6.16. Let X be a topological space and let $\mathcal{O}_{\text{top}}$ be the sheaf of continuous real-valued functions on X. Let ξ be an $\mathbb{R}$-vector bundle over X and let $\Gamma(\xi)$ be the set of all sections of ξ over X. Then $\Gamma(\xi)$ is an $\mathcal{O}_{\text{top}}$-module.

Notation 6.17. Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathbf{Mod}_{\mathcal{O}_X}$ denote the category of all $\mathcal{O}_X$-modules.

Proposition 6.18. *$\mathbf{Mod}_{\mathcal{O}_X}$ is an Abelian category.*

Definition 6.19. $\mathscr{F}$ is a *free* $\mathcal{O}_X$-module if $\mathscr{F} \cong \bigoplus \mathcal{O}_X$. That is, $\mathscr{F}(U) \cong \bigoplus \mathcal{O}_X(U)$ as an $\mathcal{O}_X(U)$-module, for each open $U \subseteq X$. $\mathscr{F}$ is *locally free* if there exists $\{U_i\}$, an open cover of X such that $\mathscr{F}|_{U_i} \cong \bigoplus \mathcal{O}_{U_i}$ for all i.

Proposition 6.20. *If $\mathscr{F}$ and $\mathscr{G}$ are locally free $\mathcal{O}_X$-modules, then $\mathscr{F} \oplus \mathscr{G}$ is a locally free $\mathcal{O}_X$-module.*

Definition 6.21. The *rank* of a locally free module $\mathscr{F}$ is defined pointwise: $\operatorname{rank}_x(\mathscr{F}) = \operatorname{rank}(\mathscr{F}|_U)$ as a free $\mathcal{O}_U$-module, where U a neighborhood of X in which $\mathscr{F}|_U$ is free.

Remark 6.22. $x \mapsto \operatorname{rank}_x(\mathscr{F})$ is locally constant and thus $\operatorname{rank}_x(\mathscr{F})$ is continuous. If X is connected then every locally free module has constant rank.

Definition 6.23. A *vector bundle* over a ringed space $(X, \mathcal{O}_X)$ is a locally free $\mathcal{O}_X$-module with $\operatorname{rank}_x(\mathscr{F}) < \infty$ for all $x \in X$.

Notation 6.24. We write $\mathbf{VB}(X, \mathcal{O}_X)$ to denote the category of vector bundles on the ringed space $(X, \mathcal{O}_X)$.

Remark 6.25. By Propositions 6.17 and 6.19, $\mathbf{VB}(X, \mathcal{O}_X)$ is an additive Abelian category.

Definition 6.26. We can thus define the Grothendieck group $K_0(X, \mathcal{O}_X)$ to be the Abelian group with one generator $[\mathscr{F}]$ for each isomorphism class of vector bundles and the relation $[\mathscr{F}] + [\mathscr{G}] = [\mathscr{F} \oplus \mathscr{G}]$ for each pair of vector bundles $\mathscr{F}$ and $\mathscr{G}$.

Proposition 6.27. *There is a categorical equivalence between $\boldsymbol{VB}(X, \mathcal{O}_{\text{top}})$ and $\boldsymbol{VB}(X)$, the category of topological vector bundles over X.*

In example 5.2.1 of [Weib], Weibel gives a one-to-one correspondence between vector bundles on $(\operatorname{Spec} R, \mathcal{O})$ and finitely generated projective R-modules. This correspondence is defined as follows: For a projective R-module, $P \mapsto \widetilde{P}$, as in our above construction. Given a vector bundle $\mathscr{F}$ on $\operatorname{Spec} R$, we have that $\mathscr{F}$ is locally free by definition. Then by the patching described in 2.5 of [Weib], we construct a projective R-module.

Definition 6.28. An $\mathcal{O}_X$-module $\mathscr{F}$ is *quasi-coherent* if there is $\{U_i\}$ an open cover of X, $U_i = \operatorname{Spec} A_i$ such that there exists an A_i-module M_i with $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for each i. $\mathscr{F}$ is *coherent* if in addition, each M_i is finitely generated.

The correspondence given above shows that every vector bundle is quasi-coherent.

Proposition 6.29. *Let $X = \operatorname{Spec} R$. Then the functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between $\boldsymbol{Mod}_R$ and $\boldsymbol{Mod}_{\mathcal{O}_X qcoh}$.*

Corollary 6.30. *Let $X = \operatorname{Spec} R$. We have an equivalence of categories between $\boldsymbol{VB}(X)$ and $\boldsymbol{P}(R)$.*

Proof. This follows from the fact that there is a one-to-one correspondence between $\mathbf{P}(R)$ and $\mathbf{VB}(X)$ which are subcategories of the equivalent categories $\mathbf{Mod}_R$ and $\mathbf{Mod}_{\mathcal{O}_X qcoh}$. ☐

We can thus conclude that for an affine scheme $(X, \mathcal{O}_X) \cong (\operatorname{Spec} R, \mathcal{O})$, we have an isomorphism of Grothendieck groups $K_0(X, \mathcal{O}_X) \cong K_0 R$.

7 K_1 of a Ring (N. Castro)

Definition 7.1. The *Whitehead group* of a ring R, denoted K_1R is given by

$$K_1R := GL(R)/GL(R)',$$

where $GL(R) = \varinjlim GL_n(R)$ and $GL(R)'$ is the commutator, or the first derived group of $GL(R)$.

Here, we will begin a discussion of $K_1\mathcal{C}$, the Whitehead group of a category $\mathcal{C}$ to give an alternative definition for K_1R.

7.1 The Loop Category

Definition 7.2. Let $\mathcal{C}$ be a category. The *loop category of* $\mathcal{C}$, denoted $\Omega(\mathcal{C})$, is the category with $\operatorname{Ob}\Omega(\mathcal{C}) = \{(A,\alpha) \mid A \in \operatorname{Ob}\mathcal{C},\ \alpha \in \operatorname{Aut} A\}$. A morphism $f \in \operatorname{Mor}_{\Omega(\mathcal{C})}((A,\alpha),(B,\beta))$ is a morphism $f \in \operatorname{Mor}_{\mathcal{C}}(A,B)$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

Definition 7.3. Let $(\mathcal{C},\perp)$ be a category with product. A *composition* on $\mathcal{C}$ is a sometimes defined binary operation $\circ : \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{C}$ such that if $A \circ B$ and $C \circ D$ are defined then so is $(A\perp C)\circ(B\perp D)$ and

$$(A\perp C)\circ(B\perp D) = (A\circ B)\perp(C\circ D).$$

Remark 7.4. If $(\mathcal{C},\perp)$ is a category with product then $(\Omega(\mathcal{C}),\perp)$ is a category with product, defined by $(A,\alpha)\perp(B,\beta) = (A\perp B, \alpha\perp\beta)$.

Definition 7.5. $K_1\mathcal{C} = K_0\Omega(\mathcal{C})$. That is, $K_1\mathcal{C}$ is an Abelian group with generators $[A,\alpha]$, $A \in \operatorname{Ob}(\mathcal{C})$ and $\alpha \in \operatorname{Aut}(\mathcal{C})$ and with relations
(1). $[A,\alpha] = [B,\beta]$ if there is an isomorphism $f \in \operatorname{Mor}_{\Omega(\mathcal{C})}((A,\alpha),(B,\beta))$.
(2). $[A,\alpha] + [B,\beta] = [A\perp B, \alpha\perp\beta]$.
(3). $[A,\alpha] + [A,\alpha'] = [A,\alpha\alpha']$.

Definition 7.6. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is *full* if $\operatorname{Mor}_{\mathcal{D}}(A,B) = \operatorname{Mor}_{\mathcal{C}}(A,B)$ for every A, $B \in \operatorname{Ob}(\mathcal{D})$.

Definition 7.7. A subcategory $\mathcal{D}$ of $(\mathcal{C},\perp)$ is *cofinal* if for every $A \in \operatorname{Ob}\mathcal{C}$ there is an $A' \in \operatorname{Ob}\mathcal{C}$ and a $B \in \operatorname{Ob} D$ such that $A\perp A' = B$.

Example 7.8. Let R be a ring and let $\mathbf{P}(R)$ denote the category of finitely generated projective modules over R. Then the subcategory $\mathbf{F}(R)$ consisting of free R-modules of finite rank is a full cofinal subcategory of $\mathbf{P}(R)$.

Proposition 7.9. *Let $(\mathcal{C}, \perp)$ be a category with product and $\mathcal{C}'$ a full cofinal subcategory. Then the inclusion $i : \mathcal{C}' \longrightarrow \mathcal{C}$ induces an isomorphism*

$$K_1 i : K_1\mathcal{C}' \longrightarrow K_1\mathcal{C}.$$

Definition 7.10. Let R be a ring. $K_1R = K_1\mathbf{P}(R)$.

Proposition 7.11. *For any ring R, $K_1R \cong GL(R)^{\text{ab}} = GL(R)/GL(R)'$, for K_1R as defined above.*

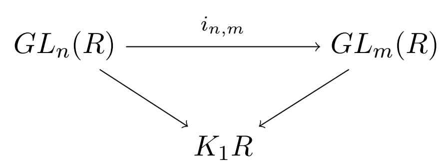
Proof. As previously stated, $GL(R) = \varinjlim GL_n(R)$. For each n, we have the homomorphism

$$i_n : GL_n(R) \longrightarrow GL_{n+1}(R)$$

via

$$M \mapsto \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}.$$

There is a map $GL_n(R) \longrightarrow K_1R$ given by $M \mapsto [R^n, M]$ such that



commutes, where $i_{n,m} = i_{m-1}i_{m-2}\cdots i_n$ $(n < m)$. Commutativity of the diagram is clear since $[R^n, M] = [R^m, M \oplus I_{m-n}]$. The universal property of the direct limit gives us a map $GL(R) \longrightarrow K_1(R)$, which induces a map $\varphi : GL(R)^{\text{ab}} \longrightarrow K_1R$.

Claim 7.12. *φ is an isomorphism.*

By the previous proposition, $K_1R = K_1\mathbf{P}(R)$, and thus every element is of the form $[R^n, M]$. We can view $M \in GL(R)$ as $M \in GL_n(R)$ for some n. So, φ maps $\overline{M} \in GL(R)^{\text{ab}}$ to $[R^n, M]$ and thus φ is surjective.

To prove injectivity, we need additional notation.

Notation 7.13. Let $B_{ij}(x) \in GL_n(R)$ be the elementary matrix which differs from I_n only in the ijth entry, where its entry is $x \in R$. The subgroup of $GL_n(R)$ generated by all such matrices is called the *elementary linear group* which we will denote $E_n(R)$.

If $i \neq j$ then

$$B_{ij}(x) = B_{ik}(x)B_{kj}(1)B_{ik}(x)^{-1}B_{kj}(1)^{-1},$$

i.e., every element of $E_n(R)$ can be written as a product of commutators. Now let $A = (a_{ij}) \in M_n(R)$. Working in $GL_{2n}(R)$ we have

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \prod_{i=1}^{n}\prod_{j=1}^{n} B_{ij+n}(a_{ij}),$$

and thus,

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \in E_{2n}(R).$$

Let $M \in GL_n(R)$. Then $M \oplus M^{-1} \in GL_{2n}(R)$, and

$$\begin{bmatrix} I & 0 \\ M^{-1}-I & I \end{bmatrix}\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ M-I & I \end{bmatrix}\begin{bmatrix} I & -M^{-1} \\ 0 & I \end{bmatrix} = M \oplus M^{-1}.$$

Thus, $M \oplus M^{-1} \in E_{2n}(R)$. Passing to $GL(R)$, $M \oplus M^{-1}$ can be written as a product of commutators and so has trivial image in $GL(R)^{\mathrm{ab}}$.

Proposition 7.14. *Let $(\mathcal{C}, \perp, \circ)$ be a category with product and composition and let A, $B \in \mathrm{Ob}\,\mathcal{C}$. Then $[A] = [B]$ in $K_1(\mathcal{C})$ if and only if there exist $C, D, E, D', E' \in \mathrm{Ob}\,\mathcal{C}$ with*

$$A \perp C \perp (D \circ E) \perp D' \perp E' = B \perp C \perp D \perp E \perp (D' \circ E').$$

Let $A \in GL_n(R)$ such that $[R^n, A] = 0$ in K_1R. Working in $\Omega(\mathbf{F}(R))$, the above proposition provides $B \in GL_s(R)$, $C_1, C_2 \in GL_t(R)$, and $D_1, D_2 \in GL_u(R)$ such that

$$(R^m, A \oplus B \oplus C_1C_2 \oplus D_1 \oplus D_2) \cong (R^r, B \oplus C_1 \oplus C_2 \oplus D_1D_2),$$

where $m = n + s + t + 2u$ and $r = s + 2t + u$. In $\Omega(\mathbf{F}(R))$,

$$(R^m, I_m) \cong (R^r, I_r).$$

Thus,

$$(R^{m+r}, A \oplus B \oplus C_1C_2 \oplus D_1 \oplus D_2 \oplus I_r) \cong (R^{m+r}, B \oplus C_1 \oplus C_2 \oplus D_1D_2 \oplus I_m).$$

Also note that

$$(R^{m+r}, I_n \oplus B^{-1} \oplus (C_1C_2)^{-1} \oplus D_1^{-1} \oplus D_2^{-1} \oplus I_r) \cong (R^{m+r}, B^{-1} \oplus (C_1C_2)^{-1} \oplus D_1^{-1} \oplus D_2^{-1} \oplus I_m).$$

Composing the above isomorphisms yields

$$(R^{m+r}, A \oplus I_s \oplus I_t \oplus I_{2u} \oplus I_r) \cong (R^{m+r}, I_s \oplus C_1 \oplus C_1^{-1} \oplus D_1 \oplus D_1^{-1} \oplus I_{m-n}).$$

Thus, A has trivial image in $GL(R)^{\mathrm{ab}}$ and so φ is injective. □

Lemma 7.15. *(Whitehead's Lemma) For any ring R, $GL(R)' \cong E(R)$.*

Thus, we have $K_1R \cong GL(R)/E(R)$.

Example 7.16. Let F be a field. Then $E(F) = SL(F)$. Consider the homomorphism

$$\det : GL(F) \longrightarrow F^{\times}.$$

Notice that $\ker(\det) = SL(F)$ and so $F^{\times} \cong GL(F)/SL(F) \cong K_1F$.

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