Twisted Homogeneous Varieties, Derived Equivalences, and dg-Stacks

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Introduction

Let X be a scheme.

- We can consider the bounded derived category $D^b(X)$ of complexes of coherent sheaves on X.
- If $\mathcal{A}$ is an ring, then we may consider the derived category $D^b(\mathcal{A})$ of complexes of finitely generated $\mathcal{A}$-modules.

There has been much work on describing the structure of the categories $D^b(X)$ and $D^b(\mathcal{A})$, and the information that can be extracted from these descriptions.

Semiorthogonal Decompositions

- Beilinson: $\mathbb{P}_{\mathbb{C}}(V)$
- Kapranov: Grassmannians and quadrics over $\mathbb{C}$
- Bernardara: Severi-Brauer schemes (étale fibrations of projective space)

HP-Duality and Categorical Representation

- Kuznetsov: Quadric fibrations and intersections of quadrics
- Auel, Bernardara, Bolognesi: Fibrations in complete intersections of quadrics

Tilting Sheaves

- Blunk, Sierra, Smith: Degree 6 del Pezzo surface
- Blunk: (Generalized) Severi-Brauer and involution varieties

Our goal is to provide derived equivalences $D^b(X) \simeq D^b(\mathcal{A})$ for schemes X which are fibrations of twisted homogeneous varieties and $\mathcal{A}$ is the endomorphism ring of an algebra on X. Our technique can be summarized as follows:

- Given a scheme X as described above with geometric fibers V, produce a tilting sheaf on V.
- Glue these sheaves to give a global sheaf $\mathcal{T}$ on X.
- Such a sheaf induces functors $\mathrm{Coh}(X) \to \mathrm{Mod}(\mathcal{A})$ and $\mathrm{Mod}(\mathcal{A}) \to \mathrm{Coh}(X)$, where $\mathcal{A}$ is the collection of endomorphisms of $\mathcal{T}$. This induces a functor on stacks associated to X and $\mathcal{A}$.
- Using the local-global nature of stacks, we can reduce to a local computation.

Contexts

Structural results of derived categories of schemes has seen a number of applications:

- Algebraic K-Theory (Quillen, Thomason-Trobaugh)
- Chow and Geometric Motives (Orlov)
- Birational geometry (Bondal-Orlov, Kuznetsov, Bolognesi-Bernardara)

K-Theory

Work of Quillen, generalized by Thomason-Trobaugh (using a generalization of higher K-theory by Waldhausen) shows that algebraic K-theory cannot distinguish rings or schemes which have equivalent derived categories.

Theorem (Quillen, Thomason-Trobaugh)

Let $\iota : \mathscr{C} \to \mathscr{D}$ be a fully faithful exact functor. If ι induces an equivalence of the bounded derived categories $D^b(\iota) : D^b(\mathscr{C}) \to D^b(\mathscr{D})$, then ι induces a homotopy equivalence of K-theory spaces

$$K(\iota) : K(\mathscr{C}) \to K(\mathscr{D}).$$

The fact that K-theory is invariant under derived equivalence allows us to compute K-groups of a scheme X via the K-groups of rings defined in terms of certain algebras on X.

Motives

D. Orlov's work on rational motives gives (under mild hypotheses) that derived equivalent projective varieties have isomorphic motives (either in the sense of Chow motives or Voevodsky's geometric motives).

Theorem (Orlov)

Let X and Y be smooth projective varieties of dimension n, and let $F : D^b(X) \to D^b(Y)$ be a fully faithful functor such that the dimension of the support of an element $a_F \in A^*(X \times Y, \mathbb{Q})$ representing F is equal to n. Then the motive $M(X)$ is a direct summand of $M(Y)$. If, in addition, F is an equivalence, then the motives $M(X)$ and $M(Y)$ are isomorphic.

Derived Eqivalences of Blunk

- Using a method for producing derived equivalences for del Pezzo surfaces of degree 6, M. Blunk has produced titling bundles for certain twisted homogenous varieties.
- We will look to extend these equivalences to fibrations of such varieties.

Tilting Sheaves

Definition

A sheaf $\mathcal{T}$ is on a smooth variety X is a *titling sheaf* if the following conditions hold:

1. The sheaf $\mathcal{T}$ has no self intersections: $R\operatorname{Hom}_{D^b(X)}(\mathcal{T}[i], \mathcal{T}) = 0$ for all $i > 0$.
2. The algebra $\operatorname{End}_{\mathcal{O}_X}(\mathcal{T})$ has finite global dimension.
3. There is no proper thick subcategory of $D^b(X)$ containing $\mathcal{T}$ (i.e. $\mathcal{T}$ generates $D^b(X)$).

Example

Let $\mathrm{SB}(A)$ be the variety of minimal right ideals of a central simple algebra A of degree n. Let $\mathcal{J}$ be the restriction of the tautological bundle on the Grassmannian $\mathrm{Gr}(n, n^2)$. Then a tilting sheaf for X is given by

$$\mathcal{T} = \mathcal{O}_X \oplus \mathcal{J} \oplus \mathcal{J}^{\otimes 2} \oplus \cdots \oplus \mathcal{J}^{\otimes (n-1)}.$$

Derived Equivalences

Theorem (Baer)

Let X be a smooth variety and let $\mathcal{T} \in \mathrm{Coh}(X)$ be a tilting sheaf with $\mathcal{A} := \mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$. Then the functors $\mathrm{Hom}(\mathcal{T}, -)$ and $(- \otimes \mathcal{T})$ induce equivalences of the corresponding derived categories

$$R\,\mathrm{Hom}(\mathcal{T}, -) : D^b(X) \to D^b(\mathcal{A})$$

$$- \otimes^L \mathcal{T} : D^b(\mathcal{A}) \to D^b(X)$$

which are inverse to each other.

M. Blunk has produced tilting sheaves in the following cases:

- Severi-Brauer varieties (twisted projective space)
- Generalized Severi-Brauer varieties (twisted Grassmannians)
- Involutions varieties (twisted projective quadrics)

K-Theory

- Blunk's derived equivalences then induce isomorphisms on K-theory.
- The computation of the K-theory of these varieties is translated to computing the K-theory of an endomorphism algebra.
- In the case of Severi-Brauer varieties, since $\mathcal{T} = \oplus \mathcal{J}^{\otimes i}$, we have a matrix representation for $\mathcal{A} = \mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$ as

$$\begin{pmatrix} \mathrm{End}(\mathcal{O}_X) & \mathrm{Hom}(\mathcal{J}, \mathcal{O}_X) & \cdots & \mathrm{Hom}(\mathcal{J}^{\otimes(n-1)}, \mathcal{O}_X) \\ 0 & \mathrm{End}(\mathcal{J}) & \cdots & \mathrm{Hom}(\mathcal{J}^{\otimes(n-1)}, \mathcal{J}) \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & \mathrm{End}(\mathcal{J}^{\otimes n-1}) \end{pmatrix}$$

- The result is an isomorphism $K_*(\mathrm{SB}(A)) \cong \bigoplus_{i=0}^{n-1} K_*(A^{\otimes i})$.

We now wish to consider similar derived equivalences for schemes which are fibrations of these twisted homogenous varieties. For example:

- Morphisms $X \to S$ which are étale-locally projective bundles over S or projective space fibrations (M. Bernardara has given a semiorthogonal decomposition for such schemes, known as *Severi-Brauer schemes*).
- Morphisms $X \to S$ which are étale-locally quadric bundles over S or projective quadric fibrations (perhaps also referred to as *involution schemes*).

Outline

- Let $X \to S$ be a fibration of twisted homogeneous varieties (for which we can produce a tilting bundle $\mathcal{T}$).
- Using techniques in cohomology of algebraic groups, show that the tilting bundles bundles on the fibers glue to give a global bundle $\mathcal{T}$ on X.
- Take $\mathcal{A} = \mathrm{End}_X(\mathcal{T})$.
- Confirm that $\mathrm{Hom}(\mathcal{T}, -)$ and $(- \otimes \mathcal{T})$ induce maps well-defined maps on the respective dg-stacks of X and $\mathcal{A}$.
- By descent (for dg-stacks), we only need to check that these functors yield an equivalence locally.
- Locally, these maps are given by tilting sheaves, so that $\mathscr{QC}^X_{\mathrm{dg}} \simeq \mathscr{QC}^{\mathcal{A}}_{\mathrm{dg}}$.
- The equivalence of dg-stacks is sufficient to conclude that the derived categories of X and $\mathcal{A}$ are equivalent ([AKW]).

dg-Categories

Definition

- A *dg-category* $\mathscr{C}$ over a commutative ring R (or an R-linear dg-category) is a category enriched in chain complexes of R-modules (i.e., morphism sets are chain complexes of R-modules), with composition given by the data: for all triples $x, y, z \in \text{ob}\, \mathscr{C}$ there is a degree 0 chain map

$$\text{Mor}_{\mathscr{C}}(y, z) \otimes_R \text{Mor}_{\mathscr{C}}(x, y) \to \text{Mor}_{\mathscr{C}}(x, z).$$

- A morphism $F : \mathscr{C} \to \mathscr{D}$ of dg-categories is a *dg-functor*, i.e., a function $F : \text{ob}(\mathscr{C}) \to \text{ob}(\mathscr{D})$ together with functorial chain maps

$$\text{Mor}_{\mathscr{C}}(x, y) \to \text{Mor}_{\mathscr{D}}(F(x), F(y)).$$

dg-Categories

Example

For an R-scheme X, let $\mathrm{QC}_{\mathrm{dg}}(X)$ denote the category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves. Then $\mathrm{QC}_{\mathrm{dg}}(X)$ is a dg-category over R.

In nice cases, the dg-category associated to a scheme encodes the same information as the bounded derived category.

Proposition (Antieau, Krashen, Ward)

If X and Y are smooth projective schemes over a field k, then $D^b(X) \simeq D^b(Y)$ as k-linear triangulated categories if and only if $\mathrm{QC}_{\mathrm{dg}}(X) \simeq \mathrm{QC}_{\mathrm{dg}}(Y)$ as dg-categories over k.

dg-Categories

- For the endomorphism ring $\mathcal{A}$ of an X-algebra, we may reformulate our above definitions and consider $\mathsf{M}_{\mathsf{dg}}(\mathcal{A})$, the associated dg-category of $\mathcal{A}$, defined as the collection of complexes of finitely generated right $\mathcal{A}$-modules.
- The functors $\mathsf{Hom}(\mathcal{T}, -)$ and $(- \otimes \mathcal{T})$ induce functors on the corresponding dg-categories

$$\mathsf{Hom}^*(\mathcal{T}, -) : \mathsf{QC}_{\mathsf{dg}}(X) \to \mathsf{M}_{\mathsf{dg}}(\mathcal{A})$$

$$(- \otimes \mathcal{T})^* : \mathsf{M}_{\mathsf{dg}}(\mathcal{A}) \to \mathsf{QC}_{\mathsf{dg}}(X).$$

Stacks

- In general, a stack $\mathscr{S}$ on a scheme S in a given topology τ is a τ-sheaf of groupoids on S, i.e., a functor $\mathscr{S} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \textbf{Grpds}$ which satisfies a certain gluing condition (descent) with respect to the topology τ.
- In fact, this data is encoded by affine schemes, so that a stack is given by an assignment of a groupoid $\mathscr{S}(\mathrm{Spec}\ R)$ for each morphism $\mathrm{Spec}\ R \to S$.
- A dg-stack $\mathscr{Q}_{\mathrm{dg}}$ is similarly defined to be an assignment of a stable presentable dg-category

$$\mathscr{Q}_{\mathrm{dg}}(\mathrm{Spec}\ R)$$

for each morphism $\mathrm{Spec}\ R \to S$, together with some gluing data.

dg-Stacks

Examples

- Let $X \to S$ be a flat, quasi-compact, quasi-separated morphism. Let $\mathscr{QC}_{\mathrm{dg}}^X$ denote the stack over S which assigns to any $\mathrm{Spec}\ R \to S$ the R-linear dg-category

$$\mathscr{QC}_{\mathrm{dg}}^X(\mathrm{Spec}\ R) = \mathrm{QC}_{\mathrm{dg}}(X_R).$$

- Similarly, we may define $\mathscr{M}_{\mathrm{dg}}^{\mathcal{A}}$ by the assignment

$$\mathscr{M}_{\mathrm{dg}}^{\mathcal{A}}(\mathrm{Spec}\ R) = \mathrm{M}_{\mathrm{dg}}(\mathcal{A}_R).$$

dg-Stacks

Using our globally defined bundle $\mathcal{T}$, we have induced maps on stacks

$$\mathscr{QC}^{X}_{\mathrm{dg}} \to \mathscr{M}^{\mathcal{A}}_{\mathrm{dg}} \qquad \mathscr{M}^{\mathcal{A}}_{\mathrm{dg}} \to \mathscr{QC}^{X}_{\mathrm{dg}}$$

defined as follows. For any morphism $\mathrm{Spec}\, R \to S$, we have a map induced by $\mathrm{Hom}(\mathcal{T}, -)$

$$\mathrm{QC}_{\mathrm{dg}}(X_R) \to \mathrm{M}_{\mathrm{dg}}(\mathcal{A}_R),$$

as well as that induced by $(- \otimes \mathcal{T})$

$$\mathrm{M}_{\mathrm{dg}}(\mathcal{A}_R) \to \mathrm{QC}_{\mathrm{dg}}(X_R).$$

It only remains to show that these functors are mutually inverse and thus define an equivalence of dg-stacks. This computation is done locally and with the help of known tilting bundles for twisted homogenous varieties.

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Thank you.