Subfields of Central Simple Algebras

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Introduction

- Central simple algebras and the Brauer group have been well studied over the past century and have seen applications to class field theory, algebraic geometry, and physics.
- Since higher K-theory defined in '72, the theory of algebraic cycles have been utilized to study geometric objects associated to central simple algebras (with involution).
- This new machinery has provided a functorial viewpoint in which to study questions of arithmetic.

Central Simple Algebras

Let F be a field.

- An F-algebra A is a ring with identity 1 such that A is an F-vector space and $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in F$ and $a, b \in A$.
- The *center* of an algebra A is

$$Z(A) = \{a \in A \mid ab = ba \text{ for every } b \in A\}.$$

Definition

A *central simple algebra* over F is an F-algebra whose only two-sided ideals are (0) and (1) and whose center is precisely F.

Examples

- An F-central division algebra, i.e., an algebra in which every element has a multiplicative inverse.
- A matrix algebra $M_n(F)$

Why Central Simple Algebras?

- Central simple algebras are a natural generalization of matrix algebras. In particular, they come equipped with a determinant (also called a *norm*).
- The collection of central simple F-algebras forms a group $\mathrm{Br}(F)$, called the *Brauer group* of F, which encodes a great deal of arithmetic structure.

Theorem (Wedderburn)

Every central simple algebra A is isomorphic to a matrix algebra with coefficients in a division algebra, i.e., there is a natural number n and a division algebra D, unique up to isomorphism, so that

$$A \cong M_n(D).$$

We thus focus only on division algebras, as every central simple algebra is "Brauer equivalent" to its underlying division algebra.

Subfields

Just as in the theory of groups, rings, etc., we aim to understand the structure of algebras based on the structure and organization of their subalgebras.

- Given a field extension $F \hookrightarrow L$, can we determine if L arises as a subfield $L \hookrightarrow D$?

Furthermore, one could ask about the degree of the field L over F relative to D.

- Is the field L a *maximal subfield* of D, i.e., $[L:F] = \sqrt{\dim(D)}$?
- Is the field L a *half-maximal subfield* of D, i.e.,

$$[L:F] = \left(\sqrt{\dim(D)}\right)^{\frac{1}{2}} ?$$

While we may not be able to answer this question in full generality, we hope to rephrase it in a "natural" way.

Maximal Subfields

- There is a naturally defined homology group $H_0(X(D), K_1)$ which parametrizes maximal subfields of an algebra D.
- To an algebra D, one can associate a geometric object $X(D)$ and consider its cohomology groups with specified coefficients.

Theorem (Merkurjev-Suslin '92)

There is a bijective correspondence between the collection of maximal subfields of a central simple algebra D and the group $(D^\times)_{\mathrm{ab}}$.

For any element $a \in D^\times$, we can associate the subfield $F(a)$, which is (generically) a maximal subfield of D.

Half-Maximal Subfields

For half-maximal subfields, the situation is a bit more subtle, so we restrict to the case where $\dim(D) = 16$.

- Assume there is a half-maximal subfield E in an algebra D. Let L be a maximal subfield containing E.
- Then $[L : F] = 4$ and $[E : F] = 2$

$$F \xrightarrow{2} E \xrightarrow{2} L$$

- If $E = F(a)$ and $L = F(b)$, then we must have $a = b^2$.
- Taking determinants, we find

$$\det(a) = \det(b^2) = \det(b)^2,$$

so a must have a square determinant.

Half-Maximal Subfields

- Indeed, this fact holds for central simple algebras whose underlying division algebra is dimension 16 (i.e., index 4).
- There is a naturally defined group $H_0(X_2(A), K_1)$ which parametrizes half-maximal subfields of an algebra A.

Theorem (M)

For an algebra A of index 4, there is bijective correspondence between the collection of half-maximal subfields of A and the collection of elements of $(A^\times)_{\mathrm{ab}}$ which have square determinant.

It is still an open question whether this holds for algebras of index p^2 for arbitrary primes p.

Thank you.