Motives Seminar

I. Overview

- Overall Goal: embed SmProj(k) into an abelian category, or at least some category w/ kernels for idempotents in a way that any Weil cohomology theory factors through the category of motives

- last time (Keller) $h: SmProj_k \to Corr(k)$
$$X \longmapsto X$$
$$f \longmapsto {}^T\Gamma_f = \{(f(x), x)\} \subseteq Y \times X\}.$$
was fully faithful, where $Corr(k) = \begin{cases} Ob = Sm, sep, f.t. \\ mor = Corr(X,Y) = CH(X \times Y) \end{cases}$

Does $h: SmProj/k \to Corr(k)$ give us what we want?
No, it doesn't have kernels for idempotents
- consider Mod_k any $e: V \to V$ s.t. $e^2 = e$ decomposes V into its 0 + 1 eigenspaces so that $V = ker(e) \oplus im(e)$.
- Since we want all abelian categories to (intuitively) act like Mod_k, we want idempotents to decompose varieties into direct sums. This doesn't happen for idempotents (Γ_e) in $Corr(X,X)$ for $X \in SmProj/k$. Need to formally add images/kernels of idempotents

II. Pseudo-abelianization

- pseudo-abelianization is a construction which formally adds in exactly the (co)kernels we need. (Note: idempotent = projector, I might use both words interchangeably)
  - strictly weaker than abelianization.

$\mathbb{F}$-linear:
$\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$,
$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Mod}_{\mathbb{F}}$

Defn Let $\mathcal{A}$ be an additive category. $\mathcal{A}$ is pseudo-abelian if $\forall A \in \mathrm{Ob}\,\mathcal{A}$ + any epic idempotent $p \in \mathrm{Hom}_{\mathcal{A}}(A,A)$, $\mathrm{Ker}(p)$ exists + $\mathrm{Ker}(p) \oplus \mathrm{Ker}(1-p) \longrightarrow A$ by $(a_1,a_2) \mapsto a_1 + a_2$ is an isomorphism.

U.P. of Pseudo-abelianization: Given an additive category $\mathcal{D}$, its pseudo-abelianization $\Psi_{\mathcal{D}} : \mathcal{D} \longrightarrow \tilde{\mathcal{D}}$ satisfies the U.P. that Ψ is a fully faithful embedding + for any $F : \mathcal{D} \longrightarrow \mathcal{B}$ additive w/ $\mathcal{B}$ pseudoabelian $\exists\, \tilde{F} : \tilde{\mathcal{D}} \longrightarrow \mathcal{B}$ s.t.

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"] \arrow[d, "\Psi_{\mathcal{D}}"'] & \mathcal{B} \\ \tilde{\mathcal{D}} \arrow[ur, "\tilde{F}"'] \arrow[u, phantom, shift left=4, "\sigma"] & \end{tikzcd}$$

i.e. F nat. iso. to $\tilde{F}\,\Psi_{\mathcal{D}}$.

All is good and well as long as $\tilde{\mathcal{D}}$ exists. Lets just say what it is + move on.

$$\mathrm{Ob}\,\tilde{\mathcal{D}} = \{(D,p) : D \in \mathrm{Ob}\,\mathcal{D},\ p = p^2 \in \mathrm{Hom}_{\mathcal{D}}(D,D)\}$$

$$\mathrm{Hom}_{\tilde{\mathcal{D}}}((D,p),(D',p')) = \frac{\{f \in \mathrm{Hom}_{\mathcal{D}}(D,D') : fp = p'f\}}{\{f \in \mathrm{Hom}_{\mathcal{D}}(D,D') : fp = p'f = 0\}}.$$

$\mathcal{D}$ additive $\Rightarrow$ the top/bottom are actually groups.

$\tilde{\mathcal{D}}$ is pseudo-abelian, + the embedding $\mathcal{D} \xrightarrow{\Psi_{\mathcal{D}}} \tilde{\mathcal{D}}$ is given by $D \longmapsto (D, 1_D)$ + $f \in \mathrm{Hom}_{\mathcal{D}}(D,D') \longmapsto [f] \in \mathrm{Hom}_{\mathcal{D}}(D,D')/(0)$.

Facts we will skip

1. if $\mathcal{D}$ is $\mathbb{F}$-linear, then $\tilde{\mathcal{D}}$ is $\mathbb{F}$-linear
2. if $\mathcal{D}$ is monoidal, so is $\tilde{\mathcal{D}}$ w/ $(X,p)\otimes(Y,q) = (X \times Y,\ p \otimes q)$.

* remember, we want our category of motives to have $\mathbb{Q}$-linear Hom sets, so at least if $\mathrm{Corr}(k)$ is $\mathbb{Q}$-linear, $\mathcal{D}$ is $\widetilde{\mathrm{Corr}(k)}$

eventually,
$M(\cdot) = \mathbb{1}$

Aside: Monoid structure on $\mathrm{Corr}(k)$:

$hX \otimes hY = h(X \times Y)$ + identity $h(\operatorname{Spec} k)$.

not writing h here

Let $f_1 \in \mathrm{Hom}_{\mathrm{Cor}_k}(X_1, Y_1)$ + $f_2 \in \mathrm{Hom}_{\mathrm{Cor}_k}(X_2, Y_2)$. Then

$$f_1 \otimes f_2 := \sigma_{23}\left(\pi_{11}^*(f_1) \cdot \pi_{22}^*(f_2)\right) \in \mathrm{Hom}_{\mathrm{cor}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$
$$= \mathrm{Corr}(X_1 \times X_2, Y_1 \times Y_2)$$
$$\cong CH(X_1 \times Y_1 \times X_2 \times Y_2)$$

Where σ_{23} switches 2^{nd} + 3^{rd} factors in the product:

$$\begin{tikzcd}
 & X_1 \times Y_1 \times X_2 \times Y_2 \arrow[dl, "\pi_{11}"'] \arrow[d, "\sigma_{23}"] \arrow[dr, "\pi_{22}"] & \\
X_1 \times Y_1 \supseteq f_1 & X_1 \times X_2 \times Y_1 \times Y_2 & X_2 \times Y_2 \supseteq f_2
\end{tikzcd}$$

Can check: $(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (f_1 \circ g_2) \otimes (f_2 \circ g_2)$. Also $h(X) \oplus h(Y) := h(X \sqcup Y)$

- So $\mathrm{Corr}(k)$ is monoidal + $\mathbb{Q}$-linear.

III. Category of pure effective Chow motives /k

- Let F = Field + char $F = 0$. usually $F = \mathbb{Q}$.
- Defn The category of effective Chow motives /k w/ coeffs in F, denoted $\mathrm{Mot}_\sim^{\mathrm{eff}}(k, F)$ is the pseudo-abelianization of $\mathrm{Corr}_\sim(k) \otimes F$. If $\sim$ = rational equiv, we've just been writing $\mathrm{Corr}(k) \otimes F$.

  If $F = \mathbb{Q}$, don't even write F.

- $\mathrm{Mot}_\sim^{\mathrm{eff}}(k)$ is $\mathbb{Q}$-linear + monoidal + pseudoabelian

Have:

$$\mathrm{SmProj}/k \xrightarrow{h} \mathrm{Corr}_\sim(k) \xrightarrow{\Psi_{\mathrm{Corr}_k}} \mathrm{Mot}_\sim^{\mathrm{eff}}(k)$$
$$X \longmapsto hX \longmapsto (hX, 1_X) = (hX, \Delta_X)$$

since $1_X = \Delta_X$ in $\mathrm{Corr}(k)$

Write M for the composition $\Psi_{\mathrm{Corr}_\sim k} \circ h$ so that $M(X)$ is the motive of X

$$\begin{array}{ccc} & \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 & \\ \swarrow & & \searrow \\ \mathbb{P}^1\times\mathbb{P}^1 & & \mathbb{P}^1\times\mathbb{P}^1 \end{array}$$

Word of caution: Manin ~~uses~~ graded correspondences w/ $Corr^r_\sim(X,Y) = C_\sim^{\dim X + r}(X\times Y)$. Thus, his defn of $Mot^{eff}_\sim(k)$ is the pseudo-abelianization of $Corr^0_\sim(k)$. $Mot^{eff}_\sim(k)$ is the same.

Note: Objects of $Mot^{eff}_\sim(k)$ look like pairs (hX, p) for $X\in SmProj/k$ + $p=p^2 \in Corr_\sim(X,X)$.

* In $Mot^{eff}_\sim k$,

$$hX = \operatorname{Ker} p \oplus \operatorname{Ker}(\Delta_X - p) = \operatorname{ker} p \oplus \operatorname{Im} p$$

so effective motives /k are essentially given by direct sum factors of sm projectives /k.

- recall, $hX\oplus hY$ is $h(X\sqcup Y)$

IV. The motive of $\mathbb{P}^1$.

- Let e be the class of a rational point, $e\in\mathbb{P}^1(k)$. e corresponds to a morphism $\operatorname{Spec} k \xhookrightarrow{p} \mathbb{P}^1$. Consider the following composition:

$$\mathbb{P}^1 \xrightarrow{\sigma} \operatorname{Spec} k \xhookrightarrow{p} \mathbb{P}^1$$

where σ is the structure morphism. $p\circ\sigma$ is a morphism $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ which gets sent by h to ${}^T\Gamma_{p\circ\sigma} \in \operatorname{Hom}_{Corr(k)}(\mathbb{P}^1,\mathbb{P}^1) = CH(\mathbb{P}^1\times\mathbb{P}^1)$.

Lets check that ${}^T\Gamma_{p\circ\sigma}$ is idempotent:

$$({}^T\Gamma_{p\circ\sigma})\circ({}^T\Gamma_{p\circ\sigma}) = {}^T(\Gamma_p\circ\Gamma_\sigma)\circ{}^T(\Gamma_p\circ\Gamma_\sigma) = {}^T\Gamma_\sigma\,{}^T\Gamma_p\,{}^T\Gamma_\sigma\,{}^T\Gamma_p$$
$$= {}^T\Gamma_\sigma\,{}^T(\Gamma_{\sigma\circ p})\,{}^T\Gamma_p = {}^T\Gamma_{p\circ\sigma}.$$

$\therefore$ ${}^T\Gamma_{p\circ\sigma}$ is idempotent. Similar calculation shows $1 - {}^T\Gamma_{p\circ\sigma}$ is also idempotent. As a cycle in $\mathbb{P}^1\times\mathbb{P}^1$

${}^T\Gamma_{p\circ\sigma} = \{(p\circ\sigma(a), a) : a\in\mathbb{P}^1\} = \{(e,a): a\in\mathbb{P}^1\} = e\times 1 \in CH(\mathbb{P}^1\times\mathbb{P}^1)$.

Claim: $\Delta - {}^T\Gamma_{p\circ\sigma} = 1\times e \in CH(\mathbb{P}^1\times\mathbb{P}^1)$. Note then

$Mot^{eff}_\sim(k) \ni (\mathbb{P}^1, {}^T\Gamma_{p\circ\sigma}) = (\operatorname{Ker}({}^T\Gamma_{p\circ\sigma}) \oplus \operatorname{Ker}(\Delta - {}^T\Gamma_{p\circ\sigma}), {}^T\Gamma_{p\circ\sigma})$

so $M(\mathbb{P}^1) = \operatorname{Ker}({}^T\Gamma_{p\circ\sigma}) \oplus \operatorname{Ker}(1 - {}^T\Gamma_{p\circ\sigma})$

$= \operatorname{Spec} k \oplus$ "other"

$=: \mathbb{1} \oplus \mathbb{L}$

$\mathbb{L}$ = Lefschetz motive

$= (\mathbb{P}^1, 1\times e)$

my notation is confusing

* $\mathbb{P}^1$ has cellular decomposition of a line + a point

More generally, $\mathcal{M}(\mathbb{P}^r) = \mathbb{1} \oplus \mathbb{L}_k \oplus \cdots \oplus \mathbb{L}_k^r$ where $\mathbb{L}_k^i = \mathbb{L}_k^{\otimes i}$

V. The category of Pure Chow Motives

- Tensoring w/ $\mathbb{L}_k$ gives us a functor

$$\otimes \mathbb{L}_k : \mathrm{Mot}_\sim^{\mathrm{eff}}(k,F) \longrightarrow \mathrm{Mot}_\sim^{\mathrm{eff}}(k,F)$$
$$\mathcal{M} \longmapsto \mathcal{M} \otimes \mathbb{L}_k$$
$$f \longmapsto f \otimes 1_{\mathbb{L}_k}$$

this functor is fully faithful $\Longrightarrow$
$\forall\, M, M' \in \mathrm{Mot}_\sim^{\mathrm{eff}}(k,F)$ + $n, m, N \in \mathbb{Z}$
$N \geq n, m$, The $\mathbb{Q}$-vector space

$$\mathrm{Hom}_{\mathrm{Mot}_\sim^{\mathrm{eff}}(k,F)}(M \otimes \mathbb{L}_k^{N-m}, M' \otimes \mathbb{L}_k^{N-n})$$

is indep of N.

Defn. Formally inverting $\mathbb{L}_k$, we get the category of pure motives $\mathrm{Mot}_\sim(k,F)$.

$$\mathrm{Ob}\,\mathrm{Mot}_\sim(k,F) = \{(M,m) : M \in \mathrm{Ob}\,\mathrm{Mot}_\sim^{\mathrm{eff}}(k,F),\ m \in \mathbb{Z}\}$$
$$\mathrm{Hom}_{\mathrm{Mot}_\sim(k,F)}((M,m),(M',n)) = \mathrm{Hom}_{\mathrm{Mot}_\sim^{\mathrm{eff}}(k,F)}(M \otimes \mathbb{L}_k^{N-m}, M' \otimes \mathbb{L}_k^{N-n})$$

where $N \geq n, m$. $\mathrm{Mot}_\sim(k) = \mathrm{Mot}_\sim(k,\mathbb{Q})$.

- concretely, objects are (X, p, n), $X \in \mathrm{SmProj}/k$,
$p = p^2 = \mathrm{End}_{\mathrm{Cor}_\sim(k,F)}(hX) = \mathrm{Mor}_{\mathrm{Cor}(k,F)}(X,X) = CH(X \times X)$, + $n \in \mathbb{Z}$.
morphisms are

$$\mathrm{Hom}_{\mathrm{Mot}_\sim}((X,p,m),(Y,q,n)) = q \circ \mathrm{Corr}_\sim^{n-m}(X,Y) \circ p$$

Denote by $\mathbb{T}_k$ the object $(\mathbb{1}_k, -1)$ + write $\mathbb{T}_k^n$ for $(\mathbb{1}_k, -n)$
So that $\mathbb{T}_k^0 = \mathbb{1}_k$ + $\mathbb{T}_k^{-1} = \mathbb{L}_k$.

$\mathbb{T}_k$ = tate motive. write $M(n) := M \otimes \mathbb{T}_k^n$ $M \in \mathcal{M} \in \mathrm{Mot}_\sim(k,F)$.
* any pure motive can be written as $M(n)$ for some n.
* for any adequate $\sim$, $\mathrm{Mot}_\sim(k)$ is $\mathbb{Q}$-linear, pseudo-abelian, tensor

$$(M,m) \otimes (M',n) = (M \otimes M', m+n).$$