Last Time

Moved motives into the noncommutative setting.

⊛ Merkurjev-Panin category $\underline{\underline{\mathcal{C}}}(k)$ of K-motives

- objects (X, A) $\quad X \in \underline{Smproj}(k)$, $A \in \underline{Sep}(k)$

- morphisms
$$\mathrm{Hom}((X,A),(Y,B)) = K_0(X \times Y, A^{op} \otimes B)$$

Computed using the category of $\mathcal{O}_{X\times Y} \otimes (A^{op} \otimes B)$-modules which are locally free $\mathcal{O}_{X\times Y}$-modules.

⊛ Tabuada's $NChow(k)$

- objects: smooth proper dg-categories $\mathcal{A}$ (up to Morita equivalence)

- $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = K_0(\mathcal{A}^{op} \otimes \mathcal{B})$.

For $X \in \underline{SmProj}(k)$, we take the dg-enhancement $D_{dg}(X)$ of the derived cat of X (defined using Drinfeld quotient) and consider the subcategory $\mathrm{perf}_{dg}(X) \subseteq D_{dg}(X)$ of perfect complexes.

$$\mathrm{Hom}\left(\mathrm{perf}_{dg}(X), \mathrm{perf}_{dg}(Y)\right) = K_0(X \times Y).$$

<u>Theorem</u> (Tabuada '11)

The assignment $X \longmapsto \mathrm{perf}_{dg}(X)$ inuces a map $\underline{SmProj}(k)^{op} \longrightarrow \underline{NChow}(k)$ which factors through Π-trivialized Chow motives

$$\underline{SmProj}(k)^{op} \longrightarrow \underline{Chow}(k) \longrightarrow \underline{Chow}(k)/_{-\otimes\Pi} \xrightarrow{\exists\,\overline{\Phi}} N\,Chow(k)$$

where $\overline{\Phi}$ is fully faithful.

How does $NChow(k)$ relate to Merk-Pan's $\underline{\underline{\mathcal{C}}}(k)$?

Theorem (Tabuada)'14. There is a fully faithful funtor

$$\Omega : \underline{\underline{\mathcal{C}}}(k) \longrightarrow NChow(k)$$

such that the maps

$$\underline{SmProj}(k)^{op} \longrightarrow NChow(k)$$
$$X \longmapsto perf_{dg}(X)$$

$$\underline{Sep}(k) \longrightarrow NChow(k)$$
$$A \longmapsto A \quad \left(\text{dg alg. concentrated in degree } 0\right)$$

factor through Ω.

(Next Page First!).

Motivic Decompositions:

Since our motives are now coming from noncommutative (derived) data, we look to

- ⊛ Semi-Orthogonal Decomp.
- ⊛ Exceptional Collections

} as correct analogues.

Let us step back: Define a category $KMot(k)$ exactly as ~~Mot(k)~~ $Chow(k)$ but replace Chow groups w/ K_0.

Objects: Smooth projective vars X

Morph: $Hom(X,Y) = K_0(X \times Y)$

$$\underline{SmProj}(k) \longrightarrow KCorr(k)$$

$$X \longmapsto X$$

$$(f : X \to Y) \longmapsto$$

Composition is defined (as usual) by by pull ⊗ push.

$\Gamma_f \subseteq X \times Y$ closed subvar.

and take $[\mathcal{O}_{\Gamma_f}] \in K_0(X \times Y)$.

Take idempotent completion $KCorr(k)^{\natural} = KMot(k)$

Thus, objects in $KMot(k)$ are pairs

(X, p) $\quad X \in \underline{SmProj}(k)$

$p \in K_0(X \times X)$ a projector.

$\underline{SmProj(k)} \longrightarrow KMot(k)$ via

$$X \longmapsto KM(X) = (X, [\mathcal{O}_{\Delta_X}]).$$

(Again, no Tate twist since no dim. grading on K_0)

Def$^{\underline{n}}$ a motive $M \in \mathrm{Chow}(k)$ is **lefschetz type**

if $M \cong \mathbb{L}^{\otimes a_1} \oplus \cdots \oplus \mathbb{L}^{\otimes a_n}$.

Def$^{\underline{n}}$ a motive $KM \in K\mathrm{Mot}(k)$ is of **unit type**

if $KM \cong KM(\mathrm{Spec}\,k)^{\oplus n}$

(we will denot $KM(\mathrm{Spec}\,k)$ by $\mathbb{1}$).

Ex: $M(\mathbb{A}^n)$ is of lefschetz type

$M(\mathbb{P}^n)$ is of ~~unit~~ lefschetz type.

$KM(\mathbb{A}^n)$ unit type

$KM(\mathbb{P}^n)$ unit type (to be shown).

<u>Def$^{\underline{n}}$</u> An object $E \in D^b(X)$ is <u>exceptional</u>

if $\quad Hom(E, E) = k$

$\quad Hom(E, E[m]) = 0$ if $m \neq 0$.

A collection $\{E_1, \dots, E_n\}$ of exceptional objects is <u>exceptional</u> if $Hom(E_j, E_i[m]) = 0$

$\forall m \in \mathbb{Z}$ when $j > i$.

(like a semi-orthonormal basis relative to $Hom(-,-)$). A collection is <u>full</u> if it generates $D^b(X)$ (smallest triangulated subcat containing collection is whole cat).

<u>Ex:</u> $\{ \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\}$ is an full exceptional collection on $\mathbb{P}^n_k$.

$$Hom(\mathcal{O}(i), \mathcal{O}(j)[m]) = Ext^m(\mathcal{O}(i), \mathcal{O}(j))$$
$$= R^m Hom(\mathcal{O}(i), \mathcal{O}(j))$$
$$= R^m (Hom(\mathcal{O}, \mathcal{O}(j-i))$$

$= R^m\Gamma(\mathbb{P}^n, \mathcal{O}(j-i))$

$$= H^m(\mathbb{P}^n, \mathcal{O}(j-i)) = \begin{cases} k & \text{if } m=0 \text{ and } i=j \\ 0 & \text{if } m \neq 0 \end{cases}$$

$0 \quad j<i$

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we write $D^b(\mathbb{P}^n) = \{\mathcal{O}, \dots, \mathcal{O}(n)\}$.

If you're a fan of semi-orthogonal decomp, the category generated by $\mathcal{O}(i)$ is $D^b(k) = D^b(\operatorname{Spec} k)$ and we have a ~~SO~~D $D^b(\mathbb{P}^n) = \langle D^b(k), \dots, D^b(k) \rangle$ $(n+1)$ copies.

<u>Rem</u>: This implies the additive invariants (mentioned before) break into direct sums.

<u>Note</u> that $M(\mathbb{P}^n) = M(\operatorname{Spec} k) \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n$.

<u>Q</u>: Is there a nice relationship between these decompositions?

For $X \in SmProj(X)$ let $\tilde{M}(X) = \bigoplus_i M(X)(i)$

<u>Theorem</u> (Orlov '05)

X, Y sm proj. vars and $F: D^b(X) \to D^b(Y)$ fully faithful. Then $\tilde{M}(X)_{\mathbb{Q}}$ is a direct summand of $\tilde{M}(Y)_{\mathbb{Q}}$. If F is an equiv. then $\tilde{M}(X) \cong \tilde{M}(Y)$.

<u>Theorem</u> (") X, Y as above, both of dimension n. Suppose F (fully faithful) is represented by object $\mathcal{A}$ on $X \times Y$ has support of dim n. Then $M(X)$ direct summand $M(y)$.

F an equivalence $\Rightarrow M(X) \cong M(y)$

Theorem (Marcolli-Tabuada) X smooth, projective k-scheme w/ $\mathrm{perf}(X) = \{E_1, \dots, E_n\}$.

If $\mathrm{char}(F) = 0$, there is a choice of integers $r_1, \dots, r_n \in \{0, \dots, \dim(X)\}$ such that

$$M(X)_F \cong \mathbb{L}^{r_1} \oplus \cdots \oplus \mathbb{L}^{r_n}$$

Chow motive. (Lefschetz type)

We can also ~~form~~ form the category $K\mathrm{Mot}(k)$ by replacing Chow groups w/ K_0.

~~___~~ Thus, a K-motive is a pair

(X, p) where $X \in \underline{\mathrm{SmProj}}(k)$

$p \in K_0(X \times X)$ a projector.

(no integer corresponding to Tate twists)

$$\mathrm{Hom}((X,p), (Y,q)) = q \circ K_0(X \times Y) \circ p.$$

let $\mathbb{1} = KM(\mathrm{Spec}\, k)$.

Q: How much ~~[illegible]~~ of Chow motive does k-motive see and vice-versa?

Theorem (Gorchinsky-Orlov)'13) $X \in \underline{SmProj}(k)$, Assume F char 0. If $M(X) \cong \mathbb{L}_F^{\otimes r_1} \oplus \cdots \oplus \mathbb{L}_F^{\otimes r_n}$ (Lefschetz type)

then $KM(X)_F = \mathbb{1}_F^{\oplus m}$

(where m = rank of $CH^*(X)$ over F).

Q: what about the converse?

$KM(X)$ unit type $\Rightarrow$ $M(X)$ Lefschetz type?

A: Generally no: The quadratic form

$\langle 1, t_1 \rangle \otimes \langle 1, t_2 \rangle \otimes \langle 1, t_3 \rangle$ over $\mathbb{Q}(t_1, t_2, t_3)$

(Tabuada 2013). Let X = corresponding quadric.

Theorem (Gorchinsky '17) X smooth proj. variety $/k$ such that $KM(X)$ is of unit type. (e.g. X admits a full exceptional collection)

If $\dim(X) \leq 2$ or $\dim(X) = 3$ and $\mathrm{char}\, k \neq 2$

Then $M(X)$ of Lefschetz type.

⊛ So far, motives (both commutative and noncommutative) look to borrow from topology via analogy (a universal cohomology theory as an analogue of singular cohomology) or topologically enriching a category (hom-complexes v.s. hom sets).

⊛ Voevodsky's approach is to ~~embed~~ completely transfer (via an embedding) algebraic geometry into the realm of algebraic topology by means of simplicial sheaves. (This is his motivic homotopy)