We defined the category $Mot_\sim(k)$ for any adequate equivalence relation $\sim$ and any field k (also along w/ a field of coeff $F = \mathbb{Q}$)

<u>Mot(k)</u>: <u>obj</u>: (X, p, n) w/ $X \in Sch/k$
p idempotent corresp.
$n \in \mathbb{Z}$ (# of Tate twists).

<u>morphisms</u>:
$$Hom_{Mot}((X,p,n),(Y,q,m)) =$$
$$q \circ CH^{\dim X - p + q}(X \times Y) \circ p$$

This solves the problem of universality for Weil Cohomology theories and this viewpoint helped solve the Weil Conjectures.

Motivating Question: What about other invariants of Schemes? Do they have a corresponding motive that controls various classes of invariants? e.g. additive or localizing invariants.

Some additive invariants:
- alg. K-theory (w $\mathbb{Z}/\ell^{\nu}$ coeffs)
- Karoubi-Villamayor K-Theory
- Nonconnective algebraic K-theory
- Homotopy K-theory
- Étale K-theory
- Mixed complex
- Cyclic Homology
- negative cyclic homology
- periodic cyclic homology
- Topological Hochschild Homology
- Topological Cyclic homology.

Tabuada ('05) constructs a "universal additive invariant" mapping to all.
This provides a framework for studying categories of motives which are noncommutative analogues of those already defined.

~~Accfunctor~~ An additive invariant is one that, for schemes, only sees the underlying (dg-enriched) derived structure, so naturally is viewed from a noncommutative perspective.

(a functor from dg-cats $\to$ additive cat
inverts Morita equivalences and takes SOD to direct sums)
(coproducts)

Preliminaries: K-theory of schemes.

Let $\mathcal{P}(X)$ denote the category of v.b. on X
(locally free sheaves = locally free $\mathcal{O}_X$-modules)

Let $K_0(X) = \mathrm{iso}(\mathcal{P}(X)) \Big/ \begin{array}{l} \text{for each SES} \\ 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \\ [\mathcal{G}] = [\mathcal{F}] + [\mathcal{H}]. \end{array}$

together w/ direct sum, this forms a group
w/ $\otimes_{\mathcal{O}_X}$ forms a ring.

<u>Notation</u>: let $\mathrm{Coh}(X)$ = category of coherent $\mathcal{O}_X$-modules on X. Same def$^{\underline{n}}$ as above is denoted $G_0(X)$

$\mathcal{P}(X)$	$\mathrm{Coh}(X)$
$K_0(X)$	$G_0(X)$
$K^0(X)$	$K_0(X)$
$K^{\bullet}(X)$	$K_{\bullet}(X)$
$K(X)$	$K'(X)$

<u>Fact</u>: These are the same if X is regular.
(Resolution theorem: every coherent sheaf has a resolution by locally frees.)

For $X = \mathrm{Spec}\, R$, $\mathcal{P}(X)$ f.g. projective modules
$M(X)$ f.g. modules.

Big Tool (holds quite a bit of generality)
like alg. spaces.

<u>Theorem</u> (Riemann-Roch)

There is a natural isomorphism of covariant functors on category of schemes of finite type over k. and proper morphisms.

$$\phi : G_0(-)_{\mathbb{Q}} \xrightarrow{\sim} CH_*(-)_{\mathbb{Q}}$$

For ~~....~~ regular schemes, this identifies

$$K_0(X)_{\mathbb{Q}} \cong CH_*(X)_{\mathbb{Q}}$$

So let's use K-theory instead of Chow theory to define motives.

<u>Merkurjev-Panin category of K-motives:</u>

let $\mathcal{C}(k)$ denote the following category

<u>obj</u>: Pairs $(X, \mathcal{A})$ $X \in Var_k$
$\mathcal{A} \in SepAlg_k$

Product of matrix algebras over division algebras w/ centers which are separable extensions of k.

<u>Morphisms</u>: $\mathrm{Hom}_{KMot}((X, \mathcal{A}), (Y, \mathcal{B}))$

$$\| $$

$$K_0(P(X \times Y, \mathcal{A}^{op} \otimes \mathcal{B}))$$

where $P(X, \mathcal{A})$ category of ~~.....~~ $(\mathcal{O}_X \otimes \mathcal{A})$-modules which are locally free as $\mathcal{O}_X$-modules.

The categories $\underline{SmProj(k)}$ and $\underline{Sep}(k)$ embed in $\underline{\underline{C}}$ via

$X \longmapsto (X, k)$ So, $\underline{\underline{C}}$ contains comm. and

$A \longmapsto (\mathrm{Spec}\, k, A)$ noncommutative data.

$\underline{\underline{C}}$ has a tensor structure

$$(X, A) \otimes (Y, B) \longmapsto (X \times Y, A \otimes B)$$

For any fixed variety V, we can define twisted K-theory

$$K_0^V : \underline{\underline{C}} \longrightarrow \underline{\underline{Ab}}$$
$$(X, A) \longmapsto (X \times V, A)$$

For any fixed separable algebra A, we can define A-twisted K-theory

$$K_0^A : \underline{\underline{C}} \longrightarrow \underline{\underline{Ab}}$$
$$(X, B) \longmapsto (X, B \otimes A).$$

Let's use this idea in a purely noncommutative (i.e. derived) setting.

$\underline{\text{Def}^n}$ A $\underline{\text{dg-algebra}}$ $/k$ is a monoid object in the category of complexes of k-modules.

$$(\mathcal{A}_\bullet \otimes \mathcal{B}_\bullet)_n = \bigoplus_{i+j=n} (\mathcal{A}_i \otimes \mathcal{B}_j)$$

That is, it's a complex $\mathcal{A}_\bullet$ together w/ a map $\mathcal{A}_\bullet \otimes \mathcal{A}_\bullet \longrightarrow \mathcal{A}_\bullet$ which respects differential of $\mathcal{A}_\bullet$.

That is, it satisfies Leibniz Rule:

$$d(ab) = d(a)b + (-1)^{\deg a} a \cdot d(b).$$

Def$^{\underline{n}}$ A dg-category $/k$ is a category enriched over dg-algebras over k (i.e. all hom sets are dg-algebras $/k$).

Ex: ⊛ Any dg-algebra is a dg-category w/ one obj.

⊛ $\mathcal{C}(k)$ = Category of complexes of k-modules. We can enrich this category by remembering the shifts. Let $C_{dg}(k)$ have same objects and

$$\mathrm{Hom}_{C_{dg}}(M, N)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(M_i, N_{i+n})$$

⊛ If $\mathcal{A}, \mathcal{B}$ are dg-cats, define $\mathcal{A} \otimes \mathcal{B}$ to be the category

obj: $\mathcal{A} \times \mathcal{B}$

morphisms: $\mathrm{Hom}((a,b),(a',b'))$

$$= \mathcal{A}(a,a') \otimes \mathcal{A}(b,b').$$

~~the shifts." let $\mathcal{C}_{dg}(k)$ be given by same objects and~~

$$\cancel{Hom_{\mathcal{C}_{dg}}(M,N)_n = \prod_{i\in\mathbb{Z}} Hom(M_i, N_{i+n})}$$

<u>dg-category associated to a scheme.</u>

X scheme $/k$ $\quad Mod(X)$ cat of $\mathcal{O}_X$-modules.

let $Ac(X) \subseteq Ch(Mod(X))$ be category of acyclic modules. let

$$Ac_{dg}(X) \subseteq \mathcal{C}_{dg}(Mod(X))$$

be the corresponding enriched categories

The quotient (of Drinfeld) $\mathcal{C}_{dg}(Mod(X))/Ac_{dg}(X)$ is the dg-cat of modules over X, $D_{dg}(X)$

(*) Can think of this as derived category w/ enrichment.

There are natural subcats

$$perf_{dg}(X) \subseteq D_{dg}(X)$$

perf. complexes (locally isomorphic to bounded complex of locally free)

$$D_{qcoh,dg}(X) \subseteq D_{dg}(X)$$

complexes of quasicoh.

Same def^{n}s apply to rings.

~~Similar construction for any ring ... using idea~~
~~categories of [illegible]~~

The category $NChow(k)$ is the idempotent completion of the category consisting of

<u>obj</u>: nice (i.e. smooth/proper) dg categories $/k$.
$\mathcal{A}$

e.g. $D_{dg}(X)$ for X smooth proper scheme $/k$.

<u>maps</u>: $Hom_{NChow}(\mathcal{A}, \mathcal{B}) = K_0(\mathcal{A}^{op} \otimes \mathcal{B})$.

on schemes $Hom_{NChow}(perf_{dg}(X), perf_{dg}(Y)) = K_0(X \times Y)$

Notice: ~~K-groups~~ K_0 doesn't reflect dimension in the same way as $\bigoplus CH^i = CH^*$, so we can't expect to have an object which shifts dimension, i.e. Tate twists.

<u>Theorem</u> (Tabuada) There is an embedding Φ

$$\begin{tikzcd} SmProj(k)^{op} \arrow[r] & Chow(k) \arrow[r] & Chow(k)/-\otimes T \arrow[d, hook, "\Phi"'] \\ & & NChow(k) \end{tikzcd}$$

as long as we trivialize tate motive, Chow motives embed into noncommutative world.