I. Recap of Mot(k)/Candace's talk

Last week, Candace defined the category of pure effective Chow motives over k.

Notation k a field (base field), F a field with $\mathrm{char}(F)=0$ (coefficient field)

$\sim$ an adequate equivalence relation

so... for example, cycles will look like $\sum n_i Z_i$

Z_i a smooth variety/k

$n_i \in F$

Def The ==category of effective chow motives== over k with coefficients in F, denoted by $\mathrm{Mot}^{\mathrm{eff}}_{\sim}(k,F)$ is the pseudoabelianization of $\mathrm{Corr}_{\sim}(k)\otimes F$

Def By formally inverting the Lefschetz motive $\mathbb{L}_k$, we get the ==category of pure motives== over k, which will write as $\mathrm{Mot}_{\sim}(k,F)$. As a reminder:

* $\mathrm{Obj}\,\mathrm{Mot}_{\sim}(k,F) = \{(M,m) \mid M \in \mathrm{Obj}\,\mathrm{Mot}^{\mathrm{eff}}_{\sim}(k,F),\ m\in\mathbb{Z}\}$
* $\mathrm{Hom}_{\mathrm{Mot}_{\sim}(k,F)}\big((M,m),(M',m')\big) = \mathrm{Hom}_{\mathrm{Mot}^{\mathrm{eff}}_{\sim}(k,L)}\big(M\otimes\mathbb{L}_k^{N-m},\ M'\otimes\mathbb{L}_k^{N-n}\big)$ with $N>n,m$.

so... objects of this category are (X,p,n) where $\begin{cases} X\in \mathrm{SmProj}(k) \\ p=p^2 \text{ (idempotent/projector)} \\ n\in\mathbb{Z}\end{cases}$

In this talk, we'll attempt to answer the following:

OUTLINE

- Why did we build Mot(k)?
- What is a Weil cohomology theory?
- Does Mot(k) give us what we want?

II. Why did we construct Mot(k)?

An issue:

Depending on the base field k we're working over, we have a lot of cohomology theories.

A few examples we'll see in this talk:

... under certain conditions, we also have:

- Betti cohomology: When k is a subfield of $\mathbb{C}$. In this case, we require $F=\mathbb{Q}$.
- Algebraic de Rham cohomology
- ℓ-adic/étale cohomology: Fix prime ℓ. Can use when $\mathrm{char}\,k\neq\ell$; $F=\mathbb{Q}_\ell$.
- Crystalline cohomology: When $\mathrm{char}(k)=\ell$.

de Rham isomorphism theorem

Artin isomorphism

These are all examples of Weil cohomology theories. (Which will be defined soon)

In algebraic topology, there are also several cohomology theories, but for reasonable spaces, they agree.

Grothendieck pictured a "universal" cohomology theory for algebraic varieties:

The theory of Motives!

There ought to be a reason for this!

There should be a "universal" cohomology theory for algebraic varieties! There ought to exist a suitable $\mathbb{Q}$-linear semisimple abelian monoidal category through which all Weil cohomology theories factor...

For reference:

a $\mathbb{Q}$-linear semisimple abelian monoidal category

Hom-sets are $\mathbb{Q}$-vector spaces, and compositions of maps are $\mathbb{Q}$-bilinear

Each object is a direct sum of finitely-many simple objects, and all such direct sums exist.

Recall: An object is simple

The category comes equipped with a monoid structure

III. Weil cohomology theories

Abstracting the properties that the previous cohomology theories share leads us to the definition of a Weil cohomology theory. Our goal for this section is to give a precise definition, and to give reasoning why the previous examples are indeed Weil cohomology theories.

Graded k-algebra

$$H^*(X) \to \bigoplus H^i(X)$$

X a smooth proj variety of dim n

Def (From Murre et al.) A Weil cohomology theory is a functor

$$H : \mathrm{SmProj}(k)^{opp} \longrightarrow \mathrm{GrVect}_F$$

char$(F) = 0$. which satisfies the following axioms

(1) There exists a cup product $\cup : H^*(X) \times H^*(X) \longrightarrow H^*(X)$ which is graded and super commutative

i.e. If $a \in H^i(X)$, $b \in H^j(X)$, then $b \cup a = (-1)^{ij} a \cup b$

(2) Poincaré duality: There exists an isomorphism $\mathrm{Tr} : H^{2n}(X) \xrightarrow{\sim} F$ such that

$$H^i(X) \times H^{2n-i}(X) \xrightarrow{\cup} H^{2n}(X) \xrightarrow[\sim]{\mathrm{Tr}} F \text{ is a perfect pairing.}$$

in particular, $H^0(\text{point}) \cong F$.

(3) Künneth formula: For X, Y in $\mathrm{SmProj}(k)$..

$$H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y) \qquad (\text{isomorphism via } P_X^* \otimes P_Y^* \ldots)$$

(4) There are cycle class maps

Algebraic cycles of codim i

$$\gamma_X : CH^i(X) \longrightarrow H^{2i}(X) \qquad \text{which are}$$

- functorial in the sense that for $f : X \to Y$ in $\mathrm{SmProj}(k)$, we have $f^* \circ \gamma_Y = \gamma_X f^*$ AND $f_* \circ \gamma_X = \gamma_Y \circ f_*$

f^* = pullback $\quad f_*$ = pushforward

- compatable with intersection product:

$$\gamma_X(\alpha . \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$$

- For points P, the following diagram commutes:

$$\begin{tikzcd} CH^0(P) \arrow[r, "\gamma_P"] \arrow[d, "\deg"] & H^0(P) \arrow[d, "\mathrm{Tr}"] \\ \mathbb{Z} \arrow[r, hook] & F \end{tikzcd}$$

(5) Weak Lefschetz: Let $Y \subseteq X$ an $n-1$ dimensional smooth hyperplane section via $i : Y \hookrightarrow X$. Then:

$$H^j(X) \xrightarrow{i^*} H^j(Y) \quad \text{is} \quad \begin{cases} \text{an isomorphism} & \text{for } j < n-1 \\ \text{injective} & \text{for } j = n-1 \end{cases}$$

(6) Hard Lefschetz: The Lefschetz operator $L(\alpha) = \alpha \cup \gamma_X(Y)$ induces isomorphisms:

$$L^{n-i} : H^{n-i}(X) \xrightarrow{\sim} H^{n+i}(X) \qquad 0 \le i \le n$$

Def Let $X \in SmProj(k)$, with $k \xhookrightarrow{\sigma} \mathbb{C}$. The **Betti cohomology** of X is the singular cohomology of X viewed in $\mathbb{C}$ via σ. (Cohomology of $\mathbb{C}$-points)

Def Let $X \in SmProj(k)$ with k algebraically closed. We have the de Rham complex:

$$\Omega_X^\bullet : \mathcal{O}_X = \Omega_X^0 \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^n \longrightarrow 0$$

We define the **algebraic de Rham cohomology** of X to be the hypercohomology of this complex.

$$H^i_{DR}(X) := \mathbf{H}^i(\Omega_X^\bullet)$$

(Recall: "Hypercohomology" → take an injective resolution $I^\bullet$ of $\Omega_X^\bullet$. Then $\mathbf{H}^i(\Omega_X^\bullet) := H^n(\Gamma(X, I^\bullet))$)

Claim Betti/singular cohomology is a Weil cohomology theory.

IV. Does Mot(k) give a "good" category of motives?

Theorem (Jannsen) $Mot(k)$ is a semisimple abelian category if (and only if) the adequate equivalence relation is numerical equivalence.

$Mot_\sim(k)$ is a $\mathbb{Q}$-linear pseudo abelian tensor category:

The tensor structure:

$$Mot_\sim(k) \times Mot_\sim(k) \xrightarrow{\otimes} Mot_\sim(k)$$

$$(X, p, m) \otimes (Y, q, n) := (X \times Y, p \times q, m+n)$$

→ Reflects the cup product axiom of Weil cohomology theory

Also, there's a duality operator:

$$Mot_\sim(k)^{opp} \xrightarrow{D} Mot_\sim(k)$$

$$M = (X, p, m) \longmapsto D(M) := (X, {}^T p, \underbrace{n}_{\dim(X)=n} - m)$$

→ much like Poincaré duality.