

I. Recap of Mot(k)/Candace's talk

Last week, Candace defined the category of pure effective Chow motives over k .

Notation k a field (basefield), F a field with $\text{char}(F)=0$ (coefficient field)

\sim an adequate equivalence relation

so... for example, cycles will look like $\sum n_i \tilde{Z}_i$
 \tilde{Z}_i a smooth variety/ k
 \uparrow
 $n_i \in F$

Def The category of effective chow motives over k with coefficients in F , denoted by $\text{Mot}_{\sim}^{\text{eff}}(k, F)$ is the pseudabelianization of $\text{Com}_{\sim}(k) \otimes F$

Def By formally inverting the Lefschetz motive \mathbb{L}_k , we get the category of pure motives over k , which we write as $\text{Mot}_{\sim}(k, F)$. As a reminder:

* $\text{Obj Mot}_{\sim}(k, F) = \{(M, m) \mid M \in \text{Obj Mot}_{\sim}^{\text{eff}}(k, F), m \in \mathbb{Z}\}$

* $\text{Hom}_{\text{Mot}_{\sim}(k, F)}((M, m), (M', m')) = \text{Hom}_{\text{Mot}_{\sim}^{\text{eff}}(k, F)}(M \otimes \mathbb{L}_k^{N-m}, M' \otimes \mathbb{L}_k^{N-m'})$ with $N > n, m$.

so... objects of this category are (X, p, n) where $\begin{cases} X \in \text{SmProj}(k) \\ p = p^2 \text{ (idempotent/projector)} \\ n \in \mathbb{Z} \end{cases}$

In this talk, we'll attempt to answer the following:

OUTLINE

- Why did we build $\text{Mot}(k)$?
- What is a Weil cohomology theory?
- Does $\text{Mot}(k)$ give us what we want?

II. Why did we construct $\text{Mot}(k)$?

An issue:

Depending on the base field k we're working over, we have a lot of cohomology theories.

A few examples we'll see in this talk:

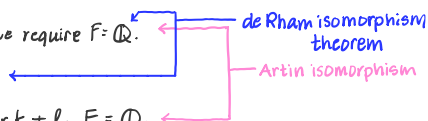
• Betti cohomology: When k is a subfield of \mathbb{C} . In this case, we require $F = \mathbb{Q}$.

• Algebraic de Rham cohomology

• l -adic/étale cohomology: Fix prime l . Can use when $\text{char } k \neq l$; $F = \mathbb{Q}_l$.

• Crystalline cohomology: When $\text{char}(k) = l$.

... under certain conditions, we also have:



These are all examples of Weil cohomology theories. (Which will be defined soon)

In algebraic topology, there are also several cohomology theories, but for reasonable spaces, they agree.

Grothendieck pictured a "universal"

cohomology theory for algebraic varieties.

The theory of Motives!

There should be a "universal" cohomology theory for algebraic varieties! There ought to exist a suitable \mathbb{Q} -linear semisimple abelian monoidal category through which all Weil cohomology theories factor...

There ought to be a reason for this!



For reference:

a \mathbb{Q} -linear semisimple abelian monoidal category

Hom-sets are \mathbb{Q} -vector spaces, and compositions of maps are \mathbb{Q} -bilinear

The category comes equipped with a monoid structure

Each object is a direct sum of finitely-many simple objects, and all such direct sums exist.

Recall: An object is simple

III. Weil cohomology theories

Abstracting the properties that the previous cohomology theories share leads us to the definition of a Weil cohomology theory. Our goal for this section is to give a precise definition, and to give reasoning why the previous examples are indeed Weil cohomology theories.

↓ Graded k -algebra

$$H^*(X) \rightarrow \bigoplus H^i(X)$$

X a smooth proj variety of dim n

Def (From Murre et al.) A **Weil cohomology theory** is a functor

$$H: \text{SmProj}(k) \xrightarrow{\text{opp}} \text{GrVect}_{\mathbb{F}}$$

$\text{char}(\mathbb{F}) \neq 0$

which satisfies the following axioms

(1) There exists a cup product $U: H^i(X) \times H^j(X) \rightarrow H^{i+j}(X)$ which is graded and super commutative
 ie. If $a \in H^i(X)$, $b \in H^j(X)$, then $b \cup a = (-1)^{ij} a \cup b$

(2) **Poincaré duality**: There exists an isomorphism $\text{Tr}: H^{2n}(X) \xrightarrow{\sim} \mathbb{F}$ such that

$$H^i(X) \times H^{2n-i}(X) \xrightarrow{U} H^{2n}(X) \xrightarrow{\sim} \mathbb{F}$$

is a perfect pairing.

in particular, $H^0(\text{point}) \cong \mathbb{F}$.

(3) **Künneth formula**: For X, Y in $\text{SmProj}(k)$..

$$H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y) \quad (\text{isomorphism via } \mathbb{P}_X^* \otimes \mathbb{P}_Y^* \dots)$$

(4) There are cycle class maps $\gamma_X: CH^i(X) \rightarrow H^{2i}(X)$ which are

$$\gamma_X: CH^i(X) \rightarrow H^{2i}(X) \quad \text{which are}$$

• functorial in the sense that for $f: X \rightarrow Y$ in $\text{SmProj}(k)$, we have $f^* \circ \gamma_Y = \gamma_X \circ f^*$ AND

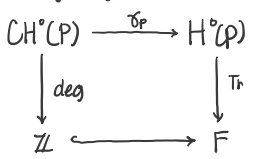
$$f_* \circ \gamma_X = \gamma_Y \circ f_*$$

f^* = pullback f_* = pushforward

• compatible with intersection product:

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$$

• For points P , the following diagram commutes:



(5) **Weak Lefschetz**: Let $Y \subseteq X$ an $n-1$ dimensional smooth hyperplane section via

$i: Y \hookrightarrow X$. Then:

$$H^j(X) \xrightarrow{i^*} H^j(Y) \quad \text{is} \quad \begin{cases} \text{an isomorphism} & \text{for } j < n-1 \\ \text{injective} & \text{for } j = n-1 \end{cases}$$

(6) **Hard Lefschetz**: The Lefschetz operator $L(\alpha) = \alpha \cup \gamma_X(Y)$ induces isomorphisms:

$$L^{n-i}: H^{n-i}(X) \xrightarrow{\sim} H^{n+i}(X) \quad 0 \leq i \leq n$$

Def Let $X \in \text{SmProj}(k)$, with $k \xrightarrow{\sigma} \mathbb{C}$. The **Betti cohomology** of X is the singular cohomology of X viewed in \mathbb{C} via σ . (Cohomology of \mathbb{C} -points)

Def Let $X \in \text{SmProj}(k)$ with k algebraically closed. We have the deRham complex:

$$\Omega_X^\bullet : \Omega_X^0 \longrightarrow \Omega_X^1 \longrightarrow \dots \longrightarrow \Omega_X^n \longrightarrow 0$$

We define the **algebraic deRham cohomology** of X to be the hypercohomology of this complex:

$$H_{\text{DR}}^i(X) := H^i(\Omega_X^\bullet) \quad \left(\begin{array}{l} \text{Recall:} \\ \text{'Hypercohomology' } \rightarrow \text{ take an injective} \\ \text{resolution } I^\bullet \text{ of } \Omega_X^\bullet. \text{ Then} \\ H^i(\Omega_X^\bullet) := H^n(\Gamma(X, I^\bullet)) \end{array} \right)$$

Claim Betti/singular cohomology is a Weil cohomology theory.

IV. Does $\text{Mot}(k)$ give a "good" category of motives?

Theorem (Jannsen) $\text{Mot}(k)$ is a semisimple abelian category if (and only if) the adequate equivalence relation is numerical equivalence.

$\text{Mot}_{\sim}(k)$ is a \mathbb{Q} -linear pseudo abelian tensor category:

The tensor structure:

$$\begin{aligned} \text{Mot}_{\sim}(k) \times \text{Mot}_{\sim}(k) &\xrightarrow{\otimes} \text{Mot}_{\sim}(k) \\ (X, p, m) \otimes (Y, q, n) &:= (X \times Y, p \times q, m+n) \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Mot}_{\sim}(k) \times \text{Mot}_{\sim}(k) \\ (X, p, m) \otimes (Y, q, n) \end{aligned}} \right\} \begin{array}{l} \text{Reflects the cup product axiom} \\ \text{of Weil cohomology theory} \end{array}$$

Also, there's a duality operator:

$$\begin{aligned} \text{Mot}_{\sim}(k)^{\text{opp}} &\xrightarrow{D} \text{Mot}_{\sim}(k) \\ M = (X, p, m) &\mapsto D(M) := (X, T_p, n-m) \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Mot}_{\sim}(k)^{\text{opp}} \\ M = (X, p, m) \end{aligned}} \right\} \begin{array}{l} \text{much like Poincaré duality.} \\ \text{dim}(X) = n \end{array}$$