The Severi-Brauer Variety Associated to a Central Simple Algebra

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July 30, 2014

History

- 1907: Wedderburn showed that CSA's are matrix rings over division algebras.
- 1932: Severi found that SB-varieties which admit a rational point are just projective space.
- 1935: Witt and Hasse discovered the connection between CSA's and SB-varieties in the case of quaternion algebras and plane conics.
- 1944: Châtelet coins the term "variété de Brauer" and develops a cohomological connection between CSA's and SB-varieties.
- In his book *Local Fields* (1962), Serre gives a complete explanation that CSA's and SB-varieties over a field k are given by one and the same cohomology set $H^1(\mathrm{Gal}(k), PGL)$.

History

- 1982: the Quillen K-theory and K-cohomology of SB-varieties is used in the proof of the Merkurjev-Suslin Theorem: the *Galois symbol* or *norm-residue homomorphism*

$$K_2(k)/n \to H^2(k, \mu_n^{\otimes 2})$$

  is an isomorphism.
- Work of Rost and Voevodsky using motivic homotopy and cohomology of *norm varieties* (a generalization of SB-varieties) led to the proof of the Bloch-Kato Conjecture: the higher norm residue homomorphisms

$$K_p(k)/n \to H^p(k, \mu_n^{\otimes p})$$

  are isomorphisms.

Conventions

- k will denote an arbitrary field.
- All algebras will be k-algebras.
- k^a will denote a fixed algebraic closure of k, and $k^s \subset k^a$ the separable closure of k.
- For a k-algebra A, let

$$Z(A) = \{x \in A : yx = xy \text{ for all } a \in A\}$$

called the **center** of A.

Central Simple Algebras

Definition

A **central simple algebra** over k is a a finite-dimensional k-algebra which is simple as a ring and which has center k.

Examples

- Any k-central division algebra D is a CSA/k.
- The algebra $M_n(k)$ of $n \times n$ matrices with entries in k is a CSA/k.
- (Hamiliton's Quaternions) Let $\mathbb{H}$ denote the $\mathbb{R}$-algebra with basis $\{1, i, j, ij\}$ which satisfy $i^2 = j^2 = -1 \quad \text{and} \quad ji = -ij$.
- (Generalized Quaternions) Let $(a, b)_k$ denote the k-algebra with basis $\{1, i, j, ij\}$ such that

$$i^2 = a \quad j^2 = b \quad \text{and} \quad ji = -ij.$$

Notice that $\mathbb{H} = (-1, -1)_k$.

Central Simple Algebras

Theorem (Wedderburn)

For an algebra A over a field k, the following conditions are equivalent:

1. *A is central simple.*
2. *If L is an algebraically closed field containing k then $A \otimes_k L \cong M_n(L)$.*
3. *There is a finite-dimensional k-central division algebra D and an integer r such that $A \cong M_r(D)$.*

Such a field L as above is called a **splitting field** of A. In fact, there is always a splitting field which is *separable*. Wedderburn's Theorem allows us to adopt the motto

"Central simple algebras are *twisted forms* of matrix algebras."

Notice that this theorem also implies that for A a CSA$/k$, the dimension of A is a square, and we define the **degree** of A to be $\sqrt{\dim_k A}$.

Grassmannians

Definition

Let V an n-dimensional k-vector space. For any integer $0 \leq m \leq n$, we define the **Grassmannian** $\mathrm{Gr}_m(V)$ to be the collection of m-dimensional subspaces in V.

Let us now realize the Grassmannian as a subset of projective space. Let $U \subset V$ be an m-dimensional subspace spanned by $\{u_1, ..., u_m\}$. We associate to U the multivector

$$u = u_1 \wedge \cdots \wedge u_m \in \Lambda^m V.$$

A different choice of basis amounts to multiplication of u by a scalar and thus we have well-defined map (the **Plücker embedding**)

$$\mathrm{Gr}_m(V) \hookrightarrow \mathbb{P}(\Lambda^m V) = \mathbb{P}^N_k.$$

The Severi-Brauer Variety of a CSA

Let A be a CSA$/F$ of dimension n^2. Among the n-dimensional subspaces of A are the right ideals $I \lhd A$, subspaces which are invariant under right multiplication by elements of A.

Definition

Let A be a central simple algebra over k. Let $\mathrm{SB}(A)$ denote the collection of right ideals of A which are n-dimensional over k, called the **Severi-Brauer variety** of A. There is an obvious inclusion into the Grassmannian

$$\mathrm{SB}(A) \hookrightarrow \mathrm{Gr}_n(A).$$

The property of being an ideal is a closed condition, i.e., given by the zeros of a polynomial. Thus $\mathrm{SB}(A)$ has the structure of a closed subvariety of $\mathrm{Gr}(n, n^2)$.

Severi-Brauer Varieties

Examples

- The variety $\mathrm{SB}(D)$ for D a division algebra has no k-points (since D has no nontrivial ideals).
- For a (generalized) quaternion algebra $A = (a, b)_k$,

  $$\mathrm{SB}(A) \cong \{ax^2 + by^2 - z^2 = 0\} \subset \mathbb{P}^2_k.$$

  This is called the *associated conic*, studied by Hasse and Witt in 1935.
- From these examples we can extract something a bit more concrete. The algebra $\mathbb{H}$ is an $\mathbb{R}$-central division algebra and thus $\mathrm{SB}(\mathbb{H})$ has no $\mathbb{R}$-rational points. Indeed, the associated conic is given by the solutions of $x^2 + y^2 = -z^2$, of which there are none over $\mathbb{R}$.

Example

- $\mathrm{SB}(M_n(k)) = \mathbb{P}_k^{n-1}$:
  We illustrate with an example. Let $n = 3$ and let

$$\mathcal{I} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

  a right ideal in $M_3(k)$. Choose any element

$$\begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{pmatrix}$$

  in $\mathcal{I}$ (with $a \neq 0$) and associate to it the point $[0 : a : 0] \in \mathbb{P}_k^2$. Of course, for any other choice of an element in $\mathcal{I}$, we have $a' = \lambda a$ for some $\lambda \in k$.

Definition

A **Severi-Brauer variety** over k is a k-variety X such that

$$X \times_{\operatorname{Spec} k} \operatorname{Spec} L \cong \mathbb{P}^n_L$$

for a finite extension L/k. Such a field is called a **splitting field** for X.

We can now adopt a second motto:

"Severi-Brauer varieties are twisted forms of projective space."

Our terminology is thus justified since $\mathrm{SB}(A)$ is a twisted form of projective space. Indeed, extending scalars to k^a (or even just k^s), A becomes isomorphic to a matrix algebra, and we have

$$\mathrm{SB}(A) \times_{\operatorname{Spec} k} \operatorname{Spec} k^a \cong \mathrm{SB}\left(A \otimes_k k^a\right) \cong \mathrm{SB}\left(M_n(k^a)\right) \cong \mathbb{P}^{n-1}_{k^a}.$$

We can do even better. Severi-Brauer varieties are quite close to being projective space. As the following theorem shows, they are a single point away.

Theorem (Châtelet, 1944)

Let X be a Severi-Brauer variety of dimension n over k. Then $X \cong \mathbb{P}_k^n$ if and only if X has a k-rational point.

Consequently, for a central simple algebra A of degree n, $\mathrm{SB}(A) \cong \mathbb{P}_k^{n-1}$ if A has a nontrivial right ideal of rank n.

Cohomological Description

In fact, this association is no fluke and we can give it a cohomological (and therefore more functorial) interpretation. This sometimes goes by the name of *Galois descent*.

For a group G and a group A with G-action, there is a first cohomology *set* $H^1(G, A)$ which consists of cocycles, i.e., maps

$$\varphi : G \longrightarrow A,$$

which satisfy $\varphi(gh) = \varphi(g)(g\varphi(h))$.

In general, if A is non-abelian, this set does not form a group. It is, however, a pointed set, pointed by the *trivial cocycle*: $\varphi(g) = e$ for all $g \in G$.

Cohomological Description

Take $G = \text{Gal}(k^s/k)$. Loosely speaking, $H^1(G, A)$ classifies objects Ω, which are twisted forms of some object Λ (i.e., $\Omega \otimes k^s \cong \Lambda$) such that $\text{Aut}(\Lambda) = A$.

Central simple algebras are twisted forms of $M_n(k)$ ($A_{k^s} \cong M_n(k^s)$) and

$$\text{Aut}(M_n(k^s)) \cong PGL_n(k^s).$$

Severi-Brauer varieties are twisted forms of projective space ($X_{k^s} \cong \mathbb{P}^{n-1}_{k^s}$) and

$$\text{Aut}(\mathbb{P}^{n-1}_{k^s}) \cong PGL_n(k^s).$$

There is a 1-1 correspondence

$$\mathbf{SB}_k^{n-1} \longleftrightarrow H^1(G, PGL_n(k^s)) \longleftrightarrow \mathbf{CSA}_k^n.$$

References

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Thank you.