Spectra: A Home for Homology

Patrick K. McFaddin

University of Georgia

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- Classical algebraic K-theory (defined by Daniel Quillen in 1973) is a functor which takes an exact category and produces an infinite loop space.
- Modern algebraic K-theory may be realized as a functor into the stable homotopy category, the homotopy category of spectra. Algebraic topologists view algebraic K-theory as a ring spectrum- a monoid object in the category of spectra.
- Spectra were first defined by E. Lima in 1960 and revisited by J. Adams in 1974 to study generalized cohomology theories.
- The theory of spectra is used in Voevodsky's work on motivic homotopy theory, as the main objects of study are homotopy classes of spectra of Nisnevich sheaves on the category of smooth schemes.

Conventions

- We will assume throughout that all spaces are pointed and all maps are basepoint preserving. When necessary, we will denote the basepoint of a space X by x_0.
- **CW** will denote the category of CW-complexes and continuous maps.
- **Spt** will denote the category of spectra and morphisms of spectra.
- For CW-complexes X and Y, we will denote by $[X, Y]$ the collection of homotopy equivalence classes of continuous maps from X to Y.
- The n^{th} homotopy group of a CW-complex X is defined to be $\pi_n(X) = [S^n, X]$.

Constructions in **CW**

- Wedge Sum:

$$X \vee Y := X \amalg Y / \{x_0\} \sim \{y_0\}$$

- Smash Product:

$$X \wedge Y := X \times Y / X \vee Y$$

- (Reduced) Suspension:

$$\Sigma X := S^1 \wedge X$$

- Loop Space:

$$\Omega X := \{f : S^1 \to X\},$$

given the compact-open topology.

- $\Sigma S^1 \cong S^2$ and in general $\Sigma^k S^n \cong S^{n+k}$
- $\pi_1(X) = \Omega X / \simeq$
- Given a map $f : X \to Y$, there are induced maps

$$\Sigma f : \Sigma X \to \Sigma Y$$

$$\Omega f : \Omega X \to \Omega Y$$

The category **CW** has a good homotopy theory.

Definition

A map $f : X \to Y$ in **CW** is a *weak equivalence* if it induces isomorphisms

$$\pi_n(X) \cong \pi_n(Y)$$

for all $n \geq 0$.

- We may form the category **HoCW** (the *homotopy category* of **CW**) by "inverting" the weak equivalences. This requires some technical machinery to make rigorous, but we can give a simple description of this category:

$$\mathrm{Ob}(\mathbf{HoCW}) = \mathrm{Ob}(\mathbf{CW}),$$

$$\mathrm{Mor}_{\mathbf{HoCW}}(X, Y) = [X, Y].$$

- The functors Ω and Σ are adjoint in **HoCW**:

$$[\Sigma X, Y] = [X, \Omega Y]$$

Definition

- A *spectrum* is a sequence of CW-complexes

$$E = \{E_0, E_1, E_2, ...\}$$

along with maps of CW-complexes

$$\Sigma E^k \to E^{k+1}.$$

- A morphism of spectra $E \to F$ is a collection of maps $f^k : E^k \to F^k$ in **CW** which commute with the structure maps.

An Ω-*spectrum* is a spectrum in which the structure maps are weak equivalences obtained from maps

$$E^k \to \Omega E^{k+1}$$

via the Ω-Σ adjunction.

Examples of Ω-Spectra

- The Sphere Spectrum:

$$\mathbb{S} := \{S^0, S^1, S^2, \ldots\}$$

- Suspension Spectrum:
  Let X be a CW-complex.

$$\Sigma^\infty X := \{X, \Sigma X, \Sigma^2 X, \ldots\}$$

  Notice that $\mathbb{S}$ is a suspension spectrum.

- (Integral) Eilenberg- MacLane Spectrum:

$$H\mathbb{Z} := \{K(0,\mathbb{Z}), K(1,\mathbb{Z}), K(2,\mathbb{Z}), \ldots\}$$

  where $K(i,\mathbb{Z})$ is a topological space satisfying

$$\pi_j(K(i,\mathbb{Z})) = \begin{cases} \mathbb{Z} & : j = i \\ 0 & : j \neq i \end{cases}$$

The assignment $X \mapsto \Sigma^\infty X$ defines a functor

$$\Sigma^\infty : \mathbf{CW} \longrightarrow \mathbf{Spt}$$

which is a categorical embedding, so that **CW** may be naturally identified with a subcategory of **Spt**. This functor has an adjoint

$$\Omega^\infty : \mathbf{Spt} \longrightarrow \mathbf{CW}$$

$$E \mapsto \varinjlim \Omega^n E^n$$

called the *underlying (infinite loop) space* of the spectrum E.

Stable Homotopy

The category **Spt** has a good homotopy theory, defined using stable homotopy groups. We first define this for CW-complexes.

Definition

Let X be a CW-complex. The n^{th} *stable homotopy group* of X is

$$\pi_n^S(X) := \varinjlim \pi_{n+k}(\Sigma^k X),$$

where this limit is taken over the maps

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$

via $[\alpha : S^n \to X] \mapsto [\Sigma\alpha : S^{n+1} \to \Sigma X]$.

Notice that the above definition may be viewed as an extension of the homotopy groups of X to that of the suspension spectrum $\Sigma^\infty X$.

Stable Homotopy

Definition

Let E be a spectrum. The n^{th} stable homotopy group of E is

$$\pi_n(E) := \varinjlim \pi_{n+k}(E^k),$$

where this limit is taken over the maps induced from the structure maps

$$\pi_n(E^0) \to \pi_{n+1}(E^1) \to \pi_{n+2}(E^2) \to \cdots$$

Of course, for $E = \Sigma^\infty X$ a suspension spectrum, we have

$$\pi_n(E) = \varinjlim \pi_{n+k}(E^k) = \varinjlim \pi_{n+k}(\Sigma^k X) = \pi_n^S(X).$$

- Just as in the case of CW-complexes, we may form the category **HoSpt**, usually called the *stable homotopy category*. This category has a well-behaved smash product (the left-derived smash product), which is somewhat involved to define.
- There is a straightforward notion of the smash product of a spectra with a CW-complex, which we will utilize below. For a spectrum E and a CW-complex X, we define $E \wedge X$ by

$$(E \wedge X)^k = E^k \wedge X.$$

We associate to a given spectrum E a reduced homology theory, given by

$$E_n(X) := \pi_n(E \wedge X).$$

- Stable homotopy is given by the sphere spectrum

$$\pi_n^S(X) = \mathbb{S}_n(X) = \pi_n(\mathbb{S} \wedge X).$$

- Reduced (integral) homology is given by the (integral) Eilenberg-MacLane spectrum

$$\widetilde{H}_n(X, \mathbb{Z}) = (H\mathbb{Z})_n(X) = \pi_n(H\mathbb{Z} \wedge X).$$

- Notice that $E_n(X)$ depends only on the homotopy type of X so that E_n may be viewed as a homology theory on **HoCW**.

Theorem (Brown Representability)

Every reduced homology theory on **CW** *has the form* $h_n(X) = \pi_n(E \wedge X)$ *for some* Ω*-spectrum* E*.*

- This theorem is usually phrased in terms of cohomology, with

$$h^n(X) = [X, E^n],$$

which coincides with the usual notion of representability. That is, contravariant functors defined by a cohomology theory on the category **HoCW** are representable in **HoSpt**.

References

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Thank you.