Milnor-Witt Groups and Homotopy Theory of Schemes

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Introduction

- Milnor-Witt groups arise as certain motivic homotopy groups. This homotopy theory mimics that of topological spaces but can be applied to schemes, with $\mathbb{A}^1$ playing the role of the interval $[0, 1]$. Unfortunately, the category of schemes is poorly behaved due to the rigidity of its objects.
- Throughout, F denotes an algebraically closed field, $\text{char}(F) \neq 2$, and $\underline{\text{Sm}}_F$ denotes category of smooth schemes over F.

Topological spaces	$\longleftrightarrow$	"Spaces" $=$ Sheaves of topological spaces on Sm_F
Homotopy groups	$\longleftrightarrow$	Homotopy sheaves of groups

Milnor K-Theory of Fields

Definition

Let F be a field. The i^{th} tensor power of F^* is given by

$$T^i(F^*) = F^* \otimes_{\mathbb{Z}} F^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^*.$$

The tensor algebra $T(F^*)$ is given by

$$T(F^*) = \bigoplus_{i \in \mathbb{N}} T^i(F^*) = \mathbb{Z} \oplus F^* \oplus (F^*)^{\otimes 2} \oplus (F^*)^{\otimes 3} \oplus \cdots$$

Notice that $T^i(F^*)$ is a group, while $T(F^*)$ has the structure of a ring:

$$T^n(F^*) \times T^m(F^*) \to T^{n+m}(F^*)$$

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots a_n \otimes b_1 \otimes \cdots \otimes b_m.$$

Milnor K-Theory of Fields

Definition

The i^{th} Milnor K-group of F is the quotient

$$K_i^M(F) = T^i(F^*)/\langle a \otimes (1-a) \mid a \in F^* \rangle.$$

The Milnor K-theory ring of F is the direct sum of the $K_i^M(F)$, i.e.,

$$K_*^M(F) = T(F^*)/\langle a \otimes (1-a) \mid a \in F^* \rangle.$$

- We denote the image of an element $a_1 \otimes \cdots \otimes a_i$ in $K_i^M(F)$ by $\{a_1, \ldots, a_i\}$.
- The relation $\{a\} \cdot \{1-a\} = \{a, 1-a\} = 0$ is known as the *Steinberg relation*.
- By definition, for elements of degree 1, we have additivity:

$$\{ab\} = \{a\} + \{b\}.$$

Milnor-Witt K-Groups of Fields

Definition

Let F be a field. The Milnor-Witt K-theory ring of F is the graded associative ring $K_*^{MW}(F)$ generated by symbols $\{a\}$ for each $a \in F^*$ in degree 1 and one symbol η of degree -1 subject to the following relations:

- (Steinberg Relation) For each $1 \neq a \in F^*$, $\{a\} \cdot \{1-a\} = 0$
- (Semi-additivity) For any $a, b \in F^*$, $\{ab\} = \{a\} + \{b\} + \eta \cdot \{a\} \cdot \{b\}$.
- For each $u \in F^*$, $\{a\} \cdot \eta = \eta \cdot \{a\}$
- $\eta \cdot (\eta \cdot \{-1\} + 2) = 0$

- The homotopic interpretation of η is closely related to the algebraic Hopf map $\eta \in [\mathbb{A}^2 \setminus \{0\}, \mathbb{P}^1]_{\mathcal{H}(F)}$.
- In algebraic topology, this Hopf map $S^3 \to S^2$ realizes S^3 as an S^1-bundle over S^2.

Milnor-Witt Sheaves

We now wish to define the sheaf $\mathbf{K}_n^{MW}$ on $\underline{\mathrm{Sm}}_F$. In fact, it suffices to know the value of this sheaf on fields.

Proposition

Let ν be a valuation on a field F with valuation ring $\mathcal{O}_\nu$ and residue field $\kappa(\nu)$. There exists a unique graded group homomorphism

$$\partial_\nu^\pi : K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(\nu))$$

which commutes with multiplication by η, is zero on units, and which satisfies

$$\partial_\nu^\pi(\{\pi\} \cdot \{u_2\} \cdots \{u_n\}) = \{\bar{u_2}\} \cdots \{\bar{u_n}\}.$$

- For a smooth irreducible scheme X and a point $x \in X$ of codimension 1, the ring $\mathcal{O}_{X,x}$ is a DVR, with valuation ν_x. Define

$$\mathbf{K}_*^{MW}(X) = \bigcap_{x \in X^{(1)}} \ker(\partial_{\nu_x}).$$

Basic Theory

F. Morel and V. Voevodsky have defined a homotopy theory on the category

$$\underline{\mathrm{Spc}}_F := \left\{ S : (\underline{\mathrm{Sm}}_F)^{\mathrm{op}} \longrightarrow \underline{\mathrm{Top}} \mid S \text{ is a sheaf} \right\}.$$

- This category is an enlargement of the category of schemes over F.
- By considering sheaves which take values in the category of $\underline{\mathrm{Top}}$, the usual topological constructions (homotopy groups, smash products, loop spaces, etc.) may be utilized to study schemes.

Basic Theory

Examples

- Any scheme X determines a space S_X using its functor of points $U \mapsto \mathrm{Hom}(U, X)$. This set determines a topological space by taking the geometric realization of simplicial set $[n] \mapsto \mathrm{Hom}(U, X)$, for $n \in \mathbb{N}$.
- The space corresponding to $\mathbb{A}^1 \setminus \{0\}$ is the *algebraic* or *Tate circle*.
- Any topological space T defines a "constant space" S_T via $U \mapsto T$ for any scheme U.
- The space corresponding to S^1 is the *simplicial circle*.

Homotopy (Sheaves of) Groups

Given a space $S : (\underline{\text{Sch}}_F)^{\text{op}} \to \underline{\text{Top}}$, the n^{th} homotopy sheaf of S is the sheaf associated to the presheaf

$$\boldsymbol{\pi}_n(S) : \underline{\text{Sch}}_F \longrightarrow \underline{\text{Grp}}$$

$$U \mapsto \pi_n(S(U)).$$

- Notice that $\boldsymbol{\pi}_n(X)$ is a sheaf of groups for $n \geq 1$ and a sheaf of abelian groups for $n \geq 2$.
- For the constant space S_T associated to a topological space T, the homotopy sheaves recover the usual homotopy groups of T.

- There is an analogously defined sheaf $\pi_n^{\mathbb{A}^1}(S)$ for each space S.
- In addition to forgetting the difference between spaces of the same (topological) homotopy type, this sheaf also forgets the difference between the spaces S and $S \times \mathbb{A}^1$, i.e., the space $\mathbb{A}^1$ is made to be contractible.

The starting point of computing these homotopy sheaves $\pi_n^{\mathbb{A}^1}(S)$ is the computation for "spheres."

Theorem (Morel)

For every integer $n \geq 2$ there is a canonical isomorphism of sheaves

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \xrightarrow{\sim} \mathbf{K}_n^{MW}.$$

References

1. A. Asok, Splitting vector bundles and $\mathbb{A}^1$-fundamental groups of higher dimensional varieties.
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4. V. Voevodsky, Nordfjordeid Lectures on Motivic Homotopy Theory.

Thanks!