# Sheaf Cohomology on a Grothendieck Site

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Fall 2012

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## Introduction

Our first goal in this talk is to motivate the definitions of a Grothendieck site and a sheaf on a site as generlizations of a topological space and a sheaf on a topological space. In particular, we hope to make clear the analogies given in the following chart:

| Topological Space $X$ | Site $\mathscr{C}$ |
|---|---|
| $U \subset X$ open subset | $U \in \mathrm{Ob}\,\mathscr{C}$ |
| $U = \bigcup U_i$ an open cover | $\{U_i \to U\}$ a cover |
| $f : X \to Y$ continuous map,<br>$f^{-1}$ takes open sets to open sets,<br>commutes with intersection | $f : \mathscr{C} \to \mathscr{D}$ morphism,<br>$f$ takes covers to covers,<br>commutes with pullbacks |
| Sheaf $F : \mathrm{Op}(X) \to \mathscr{A}$<br>contravariant functor<br>satisfies an equalizer diagram | Sheaf $F : \mathscr{C} \to \mathscr{A}$<br>contravariant functor<br>satisfies an equalizer diagram |

We will then develop some tools necessary to define cohomology groups of a sheaf on a site, laying foundations with some categorical technicalities along the way.

# 0 Sheaves on a Topological Space

**Definition 0.1.** Let $X$ be a toplogical space. A **presheaf of sets** $F$ on $X$ is the following data:

(a) for every open $U \subset X$, a set $F(U)$ called **sections** of $F$ over $U$

(b) for every $V \subset U$ open sets, a map $\rho_{UV} : F(U) \to F(V)$ called **restriction**

which satisfy the following:

(0) $F(\emptyset) = \{*\}$, the set with one element

(1) $\rho_{UU} = \mathrm{id}_{F(u)}$

(2) if $W \subset V \subset U$ then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

If $s \in F(U)$ and $V \subset U$, we will often write $s|_V$ in place of $\rho_{UV}(s)$.

**Examples.**

- The sheaf of $\mathbb{R}$-valued continuous functions on a topological space $X$, given by $F(U) = \{f : U \to \mathbb{R} | f \text{ is continuous}\}$ for $U \subset X$ open.
- The sheaf of regular functions $\mathcal{O}_X$ on a scheme $X$.
- The sheaf of holomorphic functions (or holomorphic differential forms) on a complex manifold.
- The sheaf of $C^\infty$-functions on a smooth manifold.

**Definition 0.2.** A presheaf $F$ is called a **sheaf** if the following additional conditions are satisfied:

(3) (uniqueness) if $U \subset X$ is open, $\{V_i\}$ is an open cover of $U$ (so that $U = \bigcup V_i$) and $s \in F(U)$ satisfies $s|_{V_i} = 0$ for all $i$, then $s = 0$.

(4) (gluing) if $U \subset X$ is open, $\{V_i\}$ is an open cover of $U$ and if there is a collection of sections $\{s_i\}$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all $i, j$ then there exists a section $s \in F(U)$ such that $s|_{V_i} = s_i$ for all $i$.

**Definition 0.3.** A **morphism** of sheaves (or presheaves) $\varphi : F \to G$ is a collection of morphisms $\varphi_U : F(U) \to G(U)$, for each $U \subset X$ open, which commute with the restriction maps:

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_{UV}"'] & G(U) \arrow[d, "\rho_{UV}"] \\ F(V) \arrow[r, "\varphi_V"] & G(V) \end{tikzcd}$$

We can rephrase the above definitions more categorically with an aim to generalize. Let $X$ be a topological space and let $\mathrm{Op}(X)$ denote the category whose objects are open subsets of $X$ and whose morphisms are inclusions of open sets.

- A presheaf $F$ on $X$ is a functor $F : \mathrm{Op}(X)^{\mathrm{op}} \to \mathbf{Sets}$.

- A morphism of presheaves is a natural transformation $\varphi : F \to G$.
- A sheaf is a presheaf such that for every open cover $\{U_i\}$ of $U$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

  is an equalizer diagram.

*Remark* 0.4. In general, the **equalizer** of two maps $f, g : B \rightrightarrows C$ is an object $A$ and a morphism $e : A \to B$ with $f \circ e = g \circ e$ such that the pair $(A, e)$ is universal with respect to this property (any other object which equalizes $f, g$ has a unique map to $A$ factoring through $e$). In the category of sets, the equalizer can be realized as

$$\mathrm{Eq}(f, g) = \{x \in B | f(x) = g(x)\}.$$

Even this improved definition of a sheaf relies on our ability to take intersections of open sets, but this is not possible for objects in an arbitrary category. Thus, we will have to adjust our definition in order to successfully generalize.

**Definition 0.5.** Let $\mathscr{C}$ be a category and let $f : X \to Z \leftarrow Y : g$ be a diagram in $\mathscr{C}$. The **pullback** of $f$ along $g$ (or $g$ along $f$) is the universal object $T$ making

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

commute.

In $\mathscr{C} = \mathbf{Sets}$, the pullback can be realized as $T = X \times_Z Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$. This element-wise definition actually works for any category whose objects are sets with extra structure (so-called **concrete categories**).

Notice that if $\mathscr{C} = \mathrm{Op}(X)$ and $V, W \subset U \subset X$, then the pullback of $V$ and $W$ over $U$ is given by $V \cap W$, since

$$V \times_U W = \{(x, y) \in V \times W | x = y \in U\} = V \cap W.$$

Thus, to generalize the notion of a sheaf, we use pullbacks (as opposed to intersections). We now attempt to extend our definitions to arbitrary categories.

# 1 Sites and Sheaves

## 1.1 Sites

**Definition 1.1.** Let $\mathscr{C}$ be a category. A **family of morphisms with fixed target** in $\mathscr{C}$ is given by an object $U \in \mathrm{Ob}\,\mathscr{C}$, an indexing set $I$ and for each $i \in I$, a morphism $U_i \to U$ of $\mathscr{C}$. We denote a family by $\{U_i \to U\}_{i \in I}$.

**Definition 1.2.** A **Grothendieck topology** on a category $\mathscr{C}$ is given by a set $\mathrm{Cov}(\mathscr{C})$ of families of morphisms with fixed target (called **coverings**) satsifying the following:

1. if $V \to U$ an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathscr{C})$

2. if $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ and for all $i$ $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathscr{C})$ then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathscr{C})$

3. if $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ and $V \to U$ is a morphism in $\mathscr{C}$ and $U_i \times_U V$ exists for all $i$ then $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$

A **Grothendieck site** is given by a category $\mathscr{C}$ and a choice of Grothendieck topology.

**Examples.**

- Let $X$ be a topological space and let $\mathcal{T}_X$ denote the site consisting of the category $\mathrm{Op}(X)$ and topology given by

$$\{U_i \to U\} \in \mathrm{Cov}(\mathrm{Op}(X)) \text{ if and only if } \bigcup U_i = U.$$

  Note that by definition, we allow coverings of the empty set by the empty set, as it is an open subset of $X$. However, we must also allow the **empty covering** of the empty set (i.e. taking $I = \emptyset$ and $\{U_i \to \emptyset\}_{i\in I} \in \mathrm{Cov}(\mathrm{Op}(X))$). It turns out that dissallowing such a covering still defines a site, but the sheaves on this site will, in general, be different than those on $\mathcal{T}_X$.

- Small Zariski Site. Let $X$ be a scheme. A *Zariski covering* of $X$ is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of schemes such that each $f_i$ is an open immersion and such that $X = \bigcup f_i(X_i)$ (a so-called **surjective family** of morphisms). Let $X_{\mathrm{zar}}$ be the category of open immersions into $X$ and let $\mathrm{Cov}(X_{\mathrm{zar}})$ be the set of Zariski coverings of objects in $X_{\mathrm{zar}}$.

- Big Zariski Site. Let $X$ be a scheme. As a category, the big Zariski site is the category of schemes over $X$ and coverings are Zariski coverings.

### Étale Morphisms

**Definition 1.3.** Let $X$ and $Y$ be nonsingular varieties and let $f : X \to Y$ be a morphism. Then $f$ is **étale** at $x \in X$ if the induced map $df : T_xX \to T_{f(x)}Y$ on tangent spaces is an isomorphism. We say $f$ is étale if is so at every point.

If $X$ and $Y$ are varieties over $\mathbb{C}$, we can view them as complex manifolds (using the analytic toplogy). In this case, a map $f$ is étale if it is locally an isomorphism. Thus, any covering space of a compact Riemann surface is an example of an étale morphism. The aim of using étale maps is to generalize the notion of local ismorphism so that we may *algebraically* define "small neighborhood" as we do in the complex analytic setting. More generally, a map $f : X \to Y$ of schemes is called étale if it is flat and unramified.

**Examples.**

- Étale Site. Let $X$ be a scheme. The étale site of $X$ (denoted $X_{\text{ét}}$) is given by the category $\text{Ét}/X$ of étale $X$-schemes (schemes $S$ together with an étale morphism $S \to X$) and with coverings being surjective families of morphisms in $\text{Ét}/X$.

- Big Étale Site. For a scheme $X$, the big étale site of $X$ is denoted by $X_{\text{Ét}}$ and is given by the category $\mathrm{Sch}/X$ of schemes over $X$ with coverings being surjective families of étale $X$-morphisms.

- Flat Topology. $X_{fl}$ is given by $\mathrm{Sch}/X$ with coverings being surjective families of flat $X$-morphims of finte type.

*Remark* 1.4. We have maps of topologies (defined below):

$$X_{fl} \to X_{\text{Ét}} \to X_{\text{ét}} \to X_{\text{zar}}.$$

**Definition 1.5.** A morphism $f : \mathscr{C} \to \mathscr{C}'$ of sites is a functor on the underlying categories satisfying

1. if $\{U_i \to U\} \in \mathrm{Cov}(\mathscr{C})$ then $\{f(U_i) \to f(U)\} \in \mathrm{Cov}(\mathscr{C}')$
2. if $\{U_i \to U\} \in \mathrm{Cov}(\mathscr{C})$ and $U \to V$ is a morphism in $\mathscr{C}$, then the canonical morphism

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

(given by the universal property of the pullback on the right) is an isomorphism.

**Example.** Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the corresponding sites of opens sets. We have a functor $f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$ via $U \mapsto f^{-1}(U)$. We claim that this is gives a morphism of sites.

1. $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{T}_Y) \Leftrightarrow \bigcup U_i = U$. Since the inverse image of a continuous map commutes with unions, we have $\bigcup f^{-1}(U_i) = f^{-1}(U) \Rightarrow \{f^{-1}(U_i) \to f^{-1}(U)\} \in \mathrm{Cov}(\mathcal{T}_X)$.
2. $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{T}_Y)$ and $V \to U$ a morphism in $\mathcal{T}_Y \Leftrightarrow U = \bigcup U_i$ and $V \subset U \subset Y$. Since the inverse image of a continuous map commutes with intersections, we have (by translating our notation)

$$f^{-1}(U_i \times_U V) = f^{-1}(U_i \cap V) = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U_i) \times_{f^{-1}(U)} f^{-1}(V).$$

## 1.2 Sheaves

**Definition 1.6.** Let $\mathscr{C}$ be a Grothendieck site and $\mathscr{D}$ a category. A *presheaf* on $\mathscr{C}$ with values in $\mathscr{D}$ is simply a contravariant functor $F : \mathscr{C} \to \mathscr{D}$. A **sheaf** is a presheaf such that for all coverings $\{U_i \to U\} \in \mathrm{Cov}(\mathscr{C})$ the diagram

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j)$$

is an equalizer. If $\mathscr{D}$ is an abelian category (resp. an exact category), we may rephrase this and require that the above diagram be an exact sequence (resp. an admissible exact sequence).

*Remark* 1.7. The category of sheaves of sets on $X_{\text{ét}}$ is sometimes denoted $\widetilde{X}_{\text{ét}}$ and is called the **étale topos** of $X$.

**Definition 1.8.** A **topos** is any category which is equivalent (i.e. there is a fully faithful and essentially surjective functor) to the category of sheaves of sets on a Grothendieck site.

*Remark* 1.9. Returning to the example $\mathcal{T}_X$, let $\mathcal{T}'_X$ denote the site which is the same as $\mathcal{T}_X$ except that we dissallow the empty covering of the empty set. Consider $X = \{x\}$, a singleton. Then $\mathrm{Ob}(\mathcal{T}'_X) = \{\emptyset, X\}$ and $\mathrm{Cov}(\mathcal{T}'_X) = \{\{\emptyset \to \emptyset\}, \{X \to X\}\}$. A sheaf of sets $F$ on $\mathcal{T}'_X$ is given by a set $F(X)$, a set $F(\emptyset)$ and a map $F(X) \to F(\emptyset)$. But this is the same data as a sheaf on the space $\{\emptyset, \{x\}, \{x, y\}\}$.

### Sheaf Associated to a Presheaf

Let $\mathscr{C}$ be a Grothendieck site and let $\mathbf{Pshv}(\mathscr{C})$ be the category of abelian presheaves on $\mathscr{C}$ (contravariant functors with values in $\mathbf{Ab}$, the category of abelian groups). By definition, the category $\mathbf{Shv}(\mathscr{C})$ of abelian sheaves on $\mathscr{C}$ is a full subcategory of $\mathbf{Pshv}(\mathscr{C})$ (since maps of sheaves are just maps of presheaves). Let $i : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Pshv}(\mathscr{C})$ be the inclusion (or forgetful) functor.

**Theorem 1.10.** *The functor $i$ has a left adjoint, i.e., there is a functor $\# : \mathbf{Pshv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{C})$ such that for all $F \in \mathbf{Pshv}(\mathscr{C})$ and $G \in \mathbf{Shv}(\mathscr{C})$ we have*

$$\mathrm{Hom}_{\mathbf{Shv}(\mathscr{C})}(F^{\#}, G) = \mathrm{Hom}_{\mathbf{Pshv}(\mathscr{C})}(F, iG).$$

For $F \in \mathbf{Pshv}(\mathscr{C})$, we call $F^{\#}$ the **sheaf associated to the presheaf** $F$ or the **sheafification** of $F$. It satisfies the following universal property: for any morphism $\varphi$ from a presheaf $F$ to a sheaf $G$ there is a unique map $\psi : F^{\#} \to G$ such that

$$\begin{tikzcd} & F^{\#} \arrow[dr, "\psi"] & \\ F \arrow[ur, "\#"] \arrow[rr, "\varphi"] & & G \end{tikzcd}$$

commutes. This is precisely the adjoint relationship described above.

## 2 Sheaf Cohomology

### Outline

- Let $\mathscr{C}$ be a site. Then the category $\mathbf{Shv}(\mathscr{C})$ of abelian sheaves has enough injectives so that any sheaf has an injective resolution.
- The global sections functor $\Gamma_U : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Ab}$ is left-exact.
- The failure of exactness of $\Gamma_U$ allows us to define the derived functors $R^i\Gamma_U : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Ab}$.
- We define cohomology to be given by $H^i(U, F) = R^i\Gamma_U(F)$.

To establish a cohomology theory, we need to know that our category of interest has enough injective objects. These objects are **acyclic** (i.e. have no cohomology) which allows us to "replace" the object we are studying by an injective resolution (a complex of injective objects)

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \cdots \end{tikzcd}$$

which is **quasi-isomorphic** (induces an isomorphism on homology) to the complex concentrated in one degree (our original object of interest).

**Definition 2.1.** An object $I$ of $\mathscr{C}$ is **injective** if $I \mapsto \mathrm{Hom}(A, I)$ is exact for all objects $A$ of $\mathscr{C}$.

**Definition 2.2.** An abelian category $\mathscr{C}$ is said to have **enough injectives** if for all objects $A$ of $\mathscr{C}$, there is a monomorphism of $A$ into an injective object.

## 2.1 Categorical Notions

**Definition 2.3.** A category $\mathscr{C}$ is **additive** if the following properties hold:

1. for all objects $A, B$, the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group with bilinear composition
2. finite products and sums exist
3. $\mathscr{C}$ has a **zero object** (both initial and final)

**Definition 2.4.** An additive category is **abelian** if:

Ab1. each morphism has a kernel and cokernel

Ab2. for each morphism $u$, the canonical morphism $\bar{u} : \mathrm{coIm}\, u \to \mathrm{Im}\, u$ is an isomorphism

We also include a few other properties:

Ab3. for each family $\{A_i\}_{i \in I}$, $\bigoplus_I A_i$ exists (notice the index set may be arbitrary)

Ab5. Ab3 holds and for any increasingly filtered family of of subobjects $A_i$ of $A$ and for each system of morphisms $u_i : A_i \to B$ into a fixed object $B$ such that $u_i$ is induced by $u_j$ if $A_i \subset A_j$, there exists a unique $u : \bigcup A_i \to B$ inducing the $u_i$.

(Admitting Generators) An abelian category $\mathscr{C}$ **admits generators** if there is a family of objects $\{Z_i\}_{i \in I}$ (the **family of generators**) such that for all objects $A$ of $\mathscr{C}$ and all subobjects $B$ of $A$ with $B \neq A$, there is an index $i$ and a morphism $Z_i \to A$ which *does not* factor through $B \hookrightarrow A$.

**Proposition 2.5.** *Let $\mathscr{C}$ be an abelian category satisfying Ab3. Let $\{U_i\}$ be objects of $\mathscr{C}$ and $U := \bigoplus U_i$. Then the following are equivalent:*

1. *$\{U_i\}$ are generators of $\mathscr{C}$*
2. *$U$ is a generator of $\mathscr{C}$*
3. *for each object $A$ of $\mathscr{C}$, there is an exact sequence $\bigoplus U_i \to A \to 0$ (so that $A \cong$ quotient of $\bigoplus U_i$).*

**Examples.**

- Let $R$ be a ring with unit, and let $\mathcal{M}(R)$ be the category of left $R$-modules. Then $R$ is a generator of $\mathcal{M}(R)$.
- Let $\mathscr{C}$ be a site and $U$ an object of $\mathscr{C}$. Define the presheaf $Z_U : \mathscr{C} \to \mathbf{Ab}$ via

$$V \mapsto Z_U(V) = \bigoplus_{\mathrm{Hom}_{\mathscr{C}}(U,V)} \mathbb{Z}.$$

Then the collection $\{Z_U\}_{U \in \mathscr{C}}$ is a family of generators of $\mathbf{Pshv}(\mathscr{C})$. The collection $\{Z_U^{\#}\}_{U \in \mathscr{C}}$ is a family of generators of $\mathbf{Shv}(\mathscr{C})$.

**Theorem 2.6** (Grothendieck)**.** *Let $\mathscr{C}$ be an abelian category satisfying Ab5 and which admits generators. Then $\mathscr{C}$ has enough injectives.*

**Corollary 2.7.** *Let $\mathscr{C}$ be a site. Then $\boldsymbol{Shv}(\mathscr{C})$ has enough injectives and thus every abelian sheaf has an injective resolution.*

## 2.2 Cohomology

Let $\mathscr{C}$ be a site and let $F$ be a presheaf on $\mathscr{C}$. For a fixed $U \in \mathrm{Ob}\,\mathscr{C}$ we define the **presheaf sections functor**

$$\Gamma_U^p : \mathbf{Pshv}(\mathscr{C}) \to \mathbf{Ab}$$
$$F \mapsto F(U).$$

We will use the following fact: Let $\mathscr{C}$ be a category, let $\mathscr{C}'$ be an abelian category and let $\mathrm{Hom}(\mathscr{C}, \mathscr{C}')$ be the abelian category of functors from $\mathscr{C}$ to $\mathscr{C}'$. Then a sequence $F' \to F \to F''$ in $\mathrm{Hom}(\mathscr{C}, \mathscr{C}')$ is exact if and only if $F'(U) \to F(U) \to F''(U)$ is exact in $\mathscr{C}'$ for all objects $U$ of $\mathscr{C}$ (so that exactness of functors can be checked "pointwise").

**Corollary 2.8.** *A sequence $F' \to F \to F''$ in $\mathbf{Pshv}(\mathscr{C}) = \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$ is exact if and only if $F'(U) \to F(U) \to F''(U)$ is exact in $\mathbf{Ab}$ for all objects $U$ in $\mathscr{C}$ if and only if $\Gamma_U^p(F') \to \Gamma_U^p(F) \to \Gamma_U^p(F'')$ is exact in $\mathbf{Ab}$. Thus, $\Gamma_U^p$ is an exact functor.*

We similarly define the sections functor

$$\Gamma_U : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Ab}.$$

**Proposition 2.9.** *The functor $\Gamma_U$ is left-exact.*

*Proof.* To prove this, we will make use of the fact that if a functor has a right adjoint, it is right exact. If a functor has a left adjoint, it is left exact (recall the Hom-$\otimes$ adjunction). Notice that $\Gamma_U = \Gamma_U \circ i$, where $i : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Pshv}(\mathscr{C})$ is the inclusion functor. As we saw, $\Gamma_U^p$ is exact. Since $i$ has a right adjoint ($\#$ = sheafification), $i$ is lef-exact. Thus, $\Gamma$ is left-exact. $\square$

**Definition 2.10.** Let $F$ be an abelian sheaf on a site $\mathscr{C}$. Since $\mathbf{Shv}(\mathscr{C})$ has enough injectives, we can take an injective resolution of $F$:

$$0 \to F \to I_0 \to I_1 \to \cdots$$

Applying $\Gamma_U$, for $U \in \mathrm{Ob}(\mathscr{C})$, we obtain a complex

$$0 \to \Gamma_U(I_0) \to \Gamma_U(I_1) \to \cdots$$

We define

$$H^i(U, F) = \frac{\ker\left(\Gamma_U(I_i) \to \Gamma_U(I_{i+1})\right)}{\mathrm{Im}\left(\Gamma_U(I_{i-1}) \to \Gamma_U(I_i)\right)}.$$

*Remark* 2.11. Thus, for each abelian sheaf $F$ on $X_{\text{ét}}$ and for each étale $X$-scheme $S$, we can define $H^q(S, F)$, which are usually denoted $H^q_{\text{ét}}(S, F)$ to emphasize the choice of Grothendieck topology. As mentioned above, we have a morphism of sites $\varepsilon : X_{\text{zar}} \to X_{\text{ét}}$. For each abelian sheaf $F$ on $X_{\text{ét}}$ there is a spectral sequence (called the Leray Spectral Sequence):

$$E_2^{p,q} = H^p_{\text{zar}}(S, R^q\varepsilon^*(F)) \Rightarrow H^{p+q}_{\text{ét}}(S, F).$$

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