Exceptional collections on arithmetic toric varieties
(joint with Matthew Ballard and Alexander Duncan)

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K-theory ICM Satellite Conference Workshop
Let $k$ be a field and $\bar{k}$ its algebraic closure.

**Definition**

A $k$-torus is an algebraic group $T$ over $k$ such that

$T_{\bar{k}} := T \times_{\text{Spec}(k)} \text{Spec}(\bar{k}) \cong (\bar{k}^\times)^n$.

**Examples**

- the split torus $\mathbb{G}_m^n = (k^\times)^n$
- the circle group $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$; note that $S^1_{\mathbb{C}} \cong \mathbb{C}^\times$.
- the Weil restriction $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$ (this is $\mathbb{C}^\times$ viewed as a real variety); note that $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)_{\mathbb{C}} \cong (\mathbb{C}^\times)^2$. 
An arithmetic toric variety is a (smooth, projective) normal variety together with a faithful action of a torus which has a dense open orbit. A toric variety with torus $T$ is split if $T$ is a split torus.

Example: split toric varieties

- (products of) projective space: $\mathbb{G}_m^n \subset \mathbb{A}^n \subset \mathbb{P}^n$
- if $T = \mathbb{G}_m^n$, then $X = X(\Sigma)$ for $\Sigma \subset \mathbb{Z}^n = \text{Hom}(\mathbb{G}_m, T)$
Non-split arithmetic toric varieties

Example
- the real conic \( \{(x : y : z) \in \mathbb{P}^2_{\mathbb{R}} \mid x^2 + y^2 + z^2 = 0\} = SB(\mathbb{H}) \)
- toric variety with torus \( S^1 \)
- over \( \mathbb{C} \), this is \( \mathbb{P}^1 \) with torus \( \mathbb{C}^\times \)

Example
- real projective space \( \mathbb{P}^1_{\mathbb{R}} \)
- toric variety for two distinct tori: \( \mathbb{R}^\times \) and \( S^1 \)

Example
- the Weil restriction \( R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}) \)
- toric variety with torus \( R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \)
- over \( \mathbb{C} \), this is \( \mathbb{P}^1 \times \mathbb{P}^1 \) with torus \( (\mathbb{C}^\times)^2 \)
For a $k$-scheme $X$, let $D^b(X) = D^b(coh(X))$ denote the bounded derived category of coherent sheaves on $X$.

**Definition**

An object $E$ in $D^b(X)$ is **exceptional** if

- $\text{Ext}^n(E, E) = \text{Hom}(E, E[n]) = 0$ for all $n \neq 0$, and
- $\text{End}(E)$ is a division $k$-algebra of finite dimension.

**Definition**

A sequence $E = \{E_1, \ldots, E_s\}$ of exceptional objects is an **exceptional collection** if $\text{Ext}^n(E_i, E_j) = 0$ for all $n$ whenever $i > j$.

- $E$ is **full** if it generates $D^b(X)$.
- $E$ is **strong** if $\text{Ext}^n(E_i, E_j) = 0$ for all $n \neq 0$. 
Exceptional collections

An exceptional collection \( \{E_1, \ldots, E_n\} \) on a scheme \( X \) induces an isomorphism on \( K \)-theory

\[
K_p(X) \cong \bigoplus_{i} K_p(D_i),
\]

where \( \text{End}(E_i) = D_i \).

Example

- (Beilinson) The set \( \{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\} \) is a full strong exceptional collection of line bundles on \( \mathbb{P}^n \).
- \( \mathbb{P}^1 \times \mathbb{P}^1 \) has full strong exceptional collection given by \( \{\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)\} \), where \( \mathcal{O}(i, j) = \pi_1^* \mathcal{O}(i) \otimes \pi_2^* \mathcal{O}(j) \).

Theorem (Kawamata)

Every split toric variety has a full exceptional collection.
Examples in non-split case

Example: the real conic $X = \{ x^2 + y^2 + z^2 = 0 \} \subset \mathbb{P}^2_{\mathbb{R}}$

- $X_{\mathbb{C}} \cong \mathbb{P}^1$
- $X = \text{SB}(\mathbb{H})$, where $\mathbb{H}$ denotes Hamilton’s quaternions.
- $X$ has a full strong exceptional collection $\{ \mathcal{O}_X, \mathcal{F} \}$
- $\text{End}(\mathcal{F}) \cong \mathbb{H}$ and $\mathcal{F}_{\mathbb{C}} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Example: Weil restriction $Y = R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$

- $Y_{\mathbb{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$
- $Y$ has a full strong exceptional collection $\{ \mathcal{O}_Y, \mathcal{G}, \mathcal{H} \}$
- $\text{End}(\mathcal{G}) \cong \mathbb{C}$ and $\mathcal{G}_{\mathbb{C}} \cong \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$
- $\text{End}(\mathcal{H}) \cong \mathbb{R}$ and $\mathcal{H}_{\mathbb{C}} \cong \mathcal{O}(1, 1)$
Descent of exceptional collections

Let $X$ be a scheme with an action of a finite group $G$.

**Definition**

A set of objects $E$ in $D^b(X)$ is **$G$-stable** if for all $A \in E$ and all $g \in G$ there exists $B \in E$ such that $g^* A \cong B$.

**Theorem (Ballard, Duncan, M.)**

Let $X$ be a $k$-scheme and $L/k$ a $G$-Galois extension. Then $X_L$ admits a $G$-stable exceptional collection if and only if $X$ admits an exceptional collection.

Moreover, if one collection is full/strong/of sheaves/of vector bundles, then so is the other.

Note that if $E$ is a $G$-stable exceptional collection consisting of line bundles on $X_L$, the resulting collection on $X$ may not consist of line bundles.
A question of Merkurjev and Panin

**Definition**

A \textit{G-lattice} \( M \) is a free \( \mathbb{Z} \)-module with an action of \( G \), i.e., a homomorphism \( G \to \text{GL}(M) \cong \text{GL}_n(\mathbb{Z}) \). A \( G \)-lattice is a \textit{permutation lattice} if it has a basis which is permuted by \( G \).

**Theorem (Merkurjev, Panin ’97)**

Let \( X \) be a toric variety with splitting field \( L/k \) and \( G = \text{Gal}(L/k) \). Then \( K_0(X_L) \) is a \( G \)-lattice which is a direct summand of a permutation lattice.

**Question**

Is \( K_0(X_L) \) a permutation lattice?

Notice that the existence of a full exceptional collection on \( X \) provides an affirmative answer.
Main results

Theorem

The following possess full exceptional collections of sheaves:

- Severi-Brauer varieties (Bernardara ’09)
- $dP_6$ (Blunk, Sierra, Smith ’11)
- toric surfaces (Xie, BDM ’17)
- toric Fano 3-folds (BDM)
- all forms of 43 of the 124 split toric Fano 4-folds (BDM)
- all forms of centrally symmetric toric Fano varieties (BDM; Castravet, Tevelev)
- all forms in characteristic zero of toric varieties corresponding to Weyl fans of root systems of type $A$ (BDM; Castravet, Tevelev)
Remarks on methodology

$X$ a toric variety, $X_L$ split with fan $\Sigma$

- assume $X_L$ admits an exceptional collection of line bundles
- $G = \text{Gal}(L/k)$ acts on $X_L$ so preserves $\Sigma$, and thus preserves $\Sigma(1)$
- $\Sigma(1) \leftrightarrow \text{Div}_{T_L}(X_L)$
- this induces an action of $G$ on $\text{Pic}(X_L)$ which is completely determined by the action on $\Sigma$

Thus, to check $G$-stability, suffices to check $\text{Aut}(\Sigma)$-stability.

- methods of Bondal and Uehara give stable collections for the 3-fold and 4-fold cases using the toric Frobenius
- the centrally symmetric results use an exceptional collection independently discovered by Castravet and Tevelev.
- the root system result builds on work of Castravet and Tevelev on equivariant derived categories.
¡Muchas gracias!