Introduction to K3 Surfaces

P. McFaddin

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Introduction

Definition 0.1. A *K3 surface* is a smooth two-dimensional algebraic variety which is simply connected and has trivial canonical class.

We begin unraveling this definition by defining smoothness for a variety and continue with a discussion of divisors and differentials. We will then give a few examples of K3 surfaces as well as a dimension count of various hypersurfaces in the moduli space of K3's.

1 Divisors and the Canonical Class

Throughout this talk, we will use the term *surface* to mean a smooth variety of dimension two over a fixed algebraically closed field k.

Definition 1.1. Let $Y \subset \mathbb{A}^n$ be an affine variety of dimension r. Then Y is *nonsingular* or *smooth* at $P \in Y$ if the rank of $[(\partial f_i/\partial x_j)(P)]$ is $n - r$.

1.0.1 Examples

• $Y = \{y^2 - x(x^2 - 1) = 0\} \subset \mathbb{A}^2$. Then the matrix of partial derivatives is

$$\begin{bmatrix} -3x^2 + 1 & 2y \end{bmatrix}$$

which has rank 1 for every value of x, y. Thus, Y is smooth.

• $Y = \{z^2 - x^2 - y^2 = 0\} \subset \mathbb{A}^3$. Then the matrix of partials is

$$\begin{bmatrix} -2x & -2y & 2z \end{bmatrix}$$

We notice that at $(0, 0, 0)$ this matrix has rank $< 3 - 2 = 1$ and so Y is not smooth.

We can still use this definition for projective varieties by looking at the affine patch on which the point in question lies. While it seems that this definition depends on the embedding of our varieties in projective or affine space, smoothness is actually an intrinsic property which can be characterized by considering its local rings.

Definition 1.2. Let X be a variety. A *prime divisor* on X is an irreducible closed subvariety of codimension one. A *Weil divisor* on X is a formal integral linear combination $\sum n_i D_i$ of prime divisors. We denote by $\mathrm{Div}(X)$ the free abelian group generated by prime divisors.

On a surface, the prime divisors are just smooth curves so that an arbitrary Weil divisor is of the form $\sum n_i C_i$ with $n_i \in \mathbb{Z}$ and C_i a smooth curve. Using the theory of valuation rings, we can define the divisor of a function as the set of zeros minus the set of poles (along with multiplicity). Thus, for a rational function φ we define the divisor of φ,

$$(\varphi) = \sum n_i Z_i - \sum m_i P_i$$

where Z_i is a curve for which $\varphi = 0$, φ has poles along P_i and n_i, m_i are the orders of such zeros and poles. This leads us to the notion of linear equivalence.

Definition 1.3. Let D and D' be two divisors on a surface X. Then D is *linearly equivalent to* D' (written $D \sim D'$) if $D - D'$ is *principal*. That is, if it is equal to the divisor of a function. We define $\mathrm{Cl}\, X = \mathrm{Div}\, X/\sim$.

In order to talk about the canonical class of a surface, we must make some remarks about invertible sheaves. One can think of a sheaf as a collection of functions defined over a topological space. To each open set on our surface, we associate a collection of "nice" functions on U. We also want these functions to have the property that they will glue nicely on overlaps. For example, we may consider continuous functions, differentiable functions, C^∞ differential forms, polynomials, etc.

1.0.2 Examples

• For any topological space X, we can define the sheaf of continuous functions from X to $\mathbb{R}$. Over each open set, we associate the ring $C(U)$ of continuous functions on U. If we have continuous functions f on U and g on V which agree on $U \cap V$, we can glue f and g to be a function on $U \cup V$.

• We may similarly define the sheaf of analytic functions on a complex manifold M. To each open set $U \subset M$ we associate the ring $H(U)$ of analytic functions on U. We can then define the sheaf of analytic differential forms. To each open set U we set $D(U)$ to be the module of analytic differential forms. Notice that for each open set U, $D(U)$ is a module of $H(U)$ since multiplying an analytic differential form by an analytic function yields a differential form. This example illustrates the idea that sheaves of rings and sheaves of modules generalize the notions of modules over rings.

We can define certain nice sheaves which are called invertible sheaves. An invertible sheaf is a sheaf of modules which locally looks like a free module of rank 1. Given a surface X, the set of all invertible sheaves (up to isomorphism) on X forms a group, with the group law being given by tensor product. This group is called the Picard group of X and is denoted $\mathrm{Pic}\, X$. It turns out that invertible sheaves and divisors are intimately related, as the following theorem asserts.

Theorem 1.4. *Let X be a nonsingular variety. There is a one-to-one correspondence between invertible sheaves on X modulo isomorphism and Weil divisors on X modulo linear equivalence. That is,* $\operatorname{Pic} X \cong \operatorname{Cl} X$ *as groups.*

Given a divisor D on a surface X, the associated invertible sheaf is denoted $\mathcal{O}_X(D)$. There is a special invertible sheaf on any variety called the *canonical sheaf* or *canonical bundle*, and is denoted by $\omega_X := \bigwedge^2 \Omega^1_X$. This is the sheaf which associates to each open set U, the regular differential forms on U. That is, $\omega_X(U) = \{f dx_1 \wedge dx_2 | f \,\text{is regular on}\, U\}$.

Definition 1.5. The *canonical class* K_X of X is the divisor corresponding to the canonical sheaf $\omega_X = \bigwedge^2 \Omega^1_X$. That is, $\omega_X = \mathcal{O}_X(K_X)$.

We now state a result which allows us to compute the canonical class of a surface embedded in projective space. In the examples given below, we will use this result as well as some arithmetic of divisors to determine when certain spaces have $K_X = 0$, showing that they are K3 surfaces.

Theorem 1.6. (Adjunction Formula) *Let X be a variety and let $S \subset X$ be a codimension one subvariety. If K_X is the canonical divisor on X then*

$$K_S \cong (K_X + S)|_S$$

2 Examples of K3 Surfaces

We will begin with intersections of hypersurfaces in projective space. We would like to determine the degrees of these hypersurfaces which will give a trivial canonical class on the their intersection. This will allow us to conclude that these surfaces are in fact K3's. We will repeatedly use the following fact which will not be proven.

Fact: $K_{\mathbb{P}^n} = -(n+1)H$ where $H \subset \mathbb{P}^n$ is a hyperplane.

• Let $S_d \subset \mathbb{P}^3$ be a hypersurface of degree d. Then by the adjunction formula, we have

$$K_{S_d} = (K_{\mathbb{P}^3} + S_d)|_{S_d}$$

where we consider S_d as a prime divisor of $\mathbb{P}^3$. Thus,

$$K_{S_d} = (-4 + d)H|_{S_d},$$

so that $K_{S_d} = 0$ when $d = 4$. Thus, a degree 4 hypersurface in $\mathbb{P}^3$ is a K3 surface since it has trivial canonical class.

• Let $X := S_{d_1} \cap S_{d_2} \subset \mathbb{P}^4$ be the complete intersection of two hypersurfaces of degree d_1 and d_2. We may assume that $d_i > 1$, for otherwise $S_{d_1} \cap S_{d_2} \cong \mathbb{P}^3 \cap S_{d_2}$ is just a degree d_2 surface in $\mathbb{P}^3$, which is the case above. Again, using the adjunction formula, we have

$$K_X = (K_{\mathbb{P}^4} + X)|_X.$$

Thus, we have

$$K_X = (-5 + d_1 + d_2)H|_X$$

so that $K_X = 0$ when $d_1 = 2$ and $d_2 = 3$. Hence, X is the intersection of a cubic and a quadric.

• Let $X := S_{d_1} \cap S_{d_2} \cap S_{d_3}$ be the complete intersection of three hypersurfaces of degree d_1, d_2 and d_3 respectively. Adjunction gives

$$K_X = (-6 + d_1 + d_2 + d_3)H|_X.$$

Thus, $K_X = 0$ when $d_1 = d_2 = d_3 = 2$ so that X is an intersection of 3 quadrics.

• Let $\pi : X \to \mathbb{P}^2$ be a 2:1 cover of $\mathbb{P}^2$ ramified at $\{f = 0\}$. There is a formula

$$K_X = \pi^*(K_{\mathbb{P}^2} + \frac{1}{2}R)$$

where R is the ramification divisor in $\mathbb{P}^2$. We know that $K_{\mathbb{P}^2} = -3H$ where H is a hyperplane. Thus $K_{\mathbb{P}^2} + \frac{1}{2}R = 0$ when $R \sim 6H$ so that $K_X = 0$ when $R = 6H$. Hence, f has degree 6 so is a sextic curve in $\mathbb{P}^2$. Thus, a surface which is a 2:1 cover of $\mathbb{P}^2$ ramified at a sextic is a K3 surface.

3 Polarization and the Moduli of K3 Surfaces

In describing the moduli space of K3 surfaces, we would like a second parameter to aid in our description. We consider *polarized* K3 surfaces, where a polarization L on a surface S is an ample invertible sheaf (or ample divisor) on S. In our first three examples, this ample invertible sheaf will just be $\mathcal{O}_{\mathbb{P}^3}(1)$, $\mathcal{O}_{\mathbb{P}^4}(1)$, and $\mathcal{O}_{\mathbb{P}^5}(1)$ restricted to our respective surfaces. These are simply the sheaves corresponding to the divisors $1 \cdot H$ where H is a hyperplane in $\mathbb{P}^3$, $\mathbb{P}^4$ or $\mathbb{P}^5$. We may use L along with an intersection pairing to compute the degree of our K3 surfaces.

• Let $X \subset \mathbb{P}^3$ be a degree 4 hypersurface. Then take $L = \mathcal{O}_{\mathbb{P}^3}(1)|_X = H|_X$. Thus, $L^2 = H.H.X = H.H.4H = 4$.

• Let $X \subset \mathbb{P}^4$ be the intersection of a quadric and a cubic. Then again take $L = \mathcal{O}_{\mathbb{P}^4}(1)|_X$. Then we have $L^2 = H.H.2H.3H = 6$.

• Let $X \subset \mathbb{P}^5$ be the intersection of 3 quadrics. Taking $L = \mathcal{O}_{\mathbb{P}^5}(1)|_X$ we have $L^2 = H.H.2H.2H.2H = 8$.

Dimension Count

• Let X be a hyperplane of degree 4 in $\mathbb{P}^3$. Then f is a degree 4 homogeneous polynomial in 4 variables. In general, there are $\binom{d+n}{d}$ degree d monomials in $n+1$ variables. Of course, multiplying our polynomials by a nonzero scalar will not changing our surface in projective space, so we have $\binom{d+n}{d} - 1$ monomials. Thus, in this case we have $\binom{7}{4} - 1 = 34$. We also notice that PGL(4) acts on $\mathbb{P}^3$. If $A \in \mathrm{PGL}(4)$ acts on $\mathbb{P}^3$, we will obtain a surface in $\mathbb{P}^3$ isomorphic to our original surface. Thus, we must subtract $\dim \mathrm{PGL}(4) = 4^2 - 1 = 15$ from our total count. Thus, there are 34 - 15 = 19 parameters on which our K3 surfaces of degree 4 in $\mathbb{P}^3$ depend, so the dimension of the moduli space of polarized $K3$ surfaces of degree 4 in $\mathbb{P}^3$ is 19.

• Let X be the intersection of a cubic $f_3 = 0$ and a quadric $f_2 = 0$ in $\mathbb{P}^4$. For f_3, we have $\binom{3+4}{3} - 1 = 34$ monomials and $\binom{2+4}{2} - 1 = 14$ for f_2. Consider the homogeneous ideal $\langle f_3, f_2 \rangle$. Any $f \in \langle f_3, f_2 \rangle$ will have the form $f_3 + lf_2$ for some line l. This l will eliminate 5 degrees of freedom (one for each choice of a degree one monomial). Finally, $\dim \mathrm{PGL}(5) = 5^2 - 1 = 24$, so our final count on the dimension of the moduli space of K3's of this type is $34 + (14 - 5) - 24 = 19$.

• Let X be the intersection of 3 quadrics in $\mathbb{P}^5$. For each quadric, there are $\binom{2+5}{2} - 1 = 20$ ways of defining it. Now, let us consider the homogeneous ideal $\langle f, g, h \rangle$ defining the intersection of our 3 quadrics. Notice that for any scalars $\alpha_1, ..., \alpha_6 \in k$, we have

$$\langle f + \alpha_1 g + \alpha_2 h, \alpha_3 f + g + \alpha_4 h, \alpha_5 f + \alpha_6 g + h \rangle = \langle f, g, h \rangle.$$

The α_i are six choices which we can change and not change the surface. In order to not double count, we must subtract these 6 choices from our total count. Finally, we have $\dim \mathrm{PGL}(6) = 6^2 - 1 = 35$. Thus, the dimension of the moduli space of K3 surfaces of this type is $20 + 20 + 20 - 6 - 35 = 19$.

• Let X be a 2:1 cover of $\mathbb{P}^2$ ramified at a sextic $f_6 = 0$. There are $\binom{6+2}{2} - 1 = 27$ possible monomials which can define f_6. We also have $\dim \mathrm{PGL}(3) = 3^2 - 1 = 8$. Thus, there are 27 - 8 = 19 different sextics at which $X \to \mathbb{P}^3$ may be ramified, and we conclude that the dimension of the moduli space of such K3's is 19.

References

[1] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, New York, 1977.