Grassmannians as Moduli Spaces

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Introduction

We often think of geometric objects (varieties, schemes, manifolds, etc.) as a collection of points. In speaking about the Grassmannian, we first saw that $G(k,n)$ as the set of k-dimensional subspaces in n-space. However, as we have now seen, this does not give us the full picture. We saw in our last two lectures $Gr(k,n)$ has the structure of an algebraic variety- the common zero set of a collection of polynomials. Specifically, we saw that $Gr(k,n)$ is the common zero set of the Plücker equations and can be realized as a subvariety of projective space, via the Plücker embedding. We will continue moving in this direction and eventually see that $Gr(k,n)$ is a variety which "represents" some functor

$$F : \mathbf{Sch} \to \mathbf{Sets}$$

so that $Gr(k,n)$ can be thought of in a more functorial way, helping in our realization of $G(k,n)$ as the solution to a moduli problem. We begin with a discussion of the functor of points.

1 The Functor of Points

Let $\mathscr{C}$ be a category and fix $X \in \mathrm{Ob}\,\mathscr{C}$. We define the (contravariant) functor

$$h_X : \mathscr{C} \to \mathbf{Sets}$$

via

$$Y \mapsto \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

One checks that this is in fact a functor which takes a morphism

$$f : Y \to Z$$

to

$$h_X(f) : h_X(Z) \to h_X(Y)$$

via

$$(g : Z \to X) \mapsto (g \circ f : Y \to X).$$

The functor h_X is called the *functor of points* of the object X, and the set $h_X(Y)$ is called the set of *Y-valued points of X.*

For example, consider the category **Mnfld** of manifolds. Then for any manifold X, we have

$$h_X(\{*\}) = \mathrm{Hom}(\{*\}, X) = X,$$

the set points of X.

We also notice that the association

$$X \mapsto h_X$$

gives a functor

$$h : \mathscr{C} \to \mathbf{ContFun}(\mathscr{C}, \mathbf{Sets}).$$

Recall that $\mathbf{ContFun}(\mathscr{C}, \mathbf{Sets})$ is a category in its own right, where the objects are contravariant functors $F : \mathscr{C} \to \mathbf{Sets}$ and the morphisms are natural transformations. If two functors are isomorphic we will sometimes say that they are *naturally isomorphic.* We will want to study the category $\mathbf{ContFun}(\mathscr{C}, \mathbf{Sets})$, and specifically the "nice" functors- those that are isomorphic to h_X for some object X.

Definition 1.1. A functor $F : \mathscr{C} \to \mathbf{Sets}$ is called *representable* if $F \cong h_X$ for some $X \in \mathrm{Ob}\,\mathscr{C}$ and we say that X *represents* F.

With this terminology, we would like to think of X as an object which represents the functor h_X in some meaningful way (all the data of X is encoded in h_X and vice versa). To see this, we will state the following categorical result without proof.

Lemma 1.2. (Yoneda) *If $F : \mathscr{C} \to$ Sets is any contravariant functor,*

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(h_X, h_Y).$$

In particular, if $h_X \cong h_{X'}$ then $X \cong X'$.

Thus, the functor

$$h : \mathscr{C} \to \mathbf{ContFun}(\mathscr{C}, \mathbf{Sets})$$

$$X \mapsto h_X$$

is *fully faithful*, which gives an equivalence of $\mathscr{C}$ with a full subcategory of $\mathbf{ContFun}(\mathscr{C}, \mathbf{Sets})$.

Definition 1.3. A functor $F : \mathscr{C} \to \mathscr{D}$ is *full* if for any $X, Y \in \mathrm{Ob}\,\mathscr{C}$, the map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$

is surjective. It is *faithful* if this map is injective. If F is both full and faithful, we say F is *fully faithful.*

In summation, the functor h embeds $\mathscr{C}$ in $\mathbf{ContFun}(\mathscr{C}, \mathbf{Sets})$. Philosophically, the objects of $\mathscr{C}$ are determined by their functors, and some functors are determined by objects.

Recall that a moduli functor $F : \mathbf{Var}_{\mathbb{C}} \to \mathbf{Sets}$ is a functor which (in some sense) classifies geometric objects of interest and a moduli space is the variety (if one exists) that represents this functor. Thus, our goal is to see that $Gr(k,n)$ is an object which represents a specific moduli functor. Intuitively, since we know that we have an embedding $Gr(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$, we first consider the functor of lines, which we will see is the functor which projective space represents. We will then analogously define the subspace functor and see that it too is representable, represented by the Grassmannian.

2 The Functor of Lines

The category we will want to consider is $\mathbf{Var}_K$, the category of varieties over a field K, or more specifically $\mathbf{Var}_{\mathbb{C}}$, the category of varieties over $\mathbb{C}$. Analogous to manifolds, varieties are spaces that are made up of affine patches glued together. More precisely, an n-manifold M locally looks like $\mathbb{R}^n$, which can be seen using coordinate charts. For each $p \in M$, there exists a neighborhood U of p such that U is diffeomorphic to $\mathbb{R}^n$. Likewise, for any point p in a variety V, there is a neighborhood U of p such that $U \cong Z(I)$ (the zero set of polynomials in I) for some ideal I in $k[x_1, ..., x_n]$. Varieties of this form are called *affine varieties.* A general variety is a space which is a collection of affine varieties glued together (in a coherent way). It turns out that all the information that we would like to know about varieties is contained in the category $\mathbf{AffVar}_{\mathbb{C}}$, affine varieties over $\mathbb{C}$. It also can be shown that we have an equivalence of categories

$$\mathbf{AffVar}_{\mathbb{C}}^{\mathrm{op}} \cong \mathbf{AffAlg}_{\mathbb{C}},$$

the category of finitely generated $\mathbb{C}$-algebras with no nilpotents. We can think of these as rings for the time being. Here we will say a few words about this correspondence. Let V be a variety which is the common zero set of a collection of polynomials $f_1, ..., f_k$ in n variables. To V we associate its *coordinate ring*

$$A(V) := \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k).$$

Thus, we can just as well work in $\mathbf{AffAlg}_{\mathbb{C}}$ to analyze our functors. Notice that the coordinate ring of the point $\{(0, ..., 0)\}$ is $\mathbb{C}[x_1, ..., x_n]/(x_1, ..., x_n) \cong \mathbb{C}$ since all the polynomials in $(x_1, ..., x_n)$ vanish at this point. That is, $\mathbb{C}$ corresponds to the one-point variety so that for any variety V,

$$h_V(\mathbb{C}) = \mathrm{Hom}(\{0\}, V) \cong V$$

as sets.

Returning to our original definition of projective space, we fix a positive integer n and defined $\mathbb{P}^{n-1} = \{1\text{-dimensional subspaces of } \mathbb{C}^n\}$. But this does not tell us the whole story, since this only tells us what $\mathbb{P}^{n-1}$ is as a set. That is, this only gives us the $\mathbb{C}$-valued points. For if we consider the functor $h_{\mathbb{P}^n}$, we have only considered the set

$$h_{\mathbb{P}^n}(\mathbb{C}) = \mathbb{P}^n$$

as we saw above. We need to see how things vary in families, and to do this, we need to consider R-valued points of $\mathbb{P}^n$ for any ring R, not just $\mathbb{C}$-valued points. That is, study $\mathrm{Hom}(-, \mathbb{P}^n)$ by looking at other maps into projective space. It may be beneficial to keep in mind that for a variety defined by some collection of polynomials, $\mathbb{C}$-valued points give solutions to these polynomials over $\mathbb{C}$, while R-valued points correspond to solutions in R. We have

$$\{\mathbb{C} - \text{valued pts}\} \longleftrightarrow \{\text{Subspaces of } \mathbb{C}^n\}.$$

We now want to consider to what R-valued points should correspond. Perhaps

$$\{R - \text{valued pts}\} \longleftrightarrow \{\text{Submodules of } R^n\}.$$

Unfortunately, this is not quite right. These need to be submodules which "have a complement."

Definition 2.1. Let R be a ring. An R-module P is called *projective* if there exists an R-module Q with $P \oplus Q \cong R^n$ for some $n \in \mathbb{Z}$.

So, a projective module is not free (isomorphic to R^n for some n) but it is close to free- it is a direct summand of a free module. In this sense, a projective module is a generalization of a free module. To define our functor of lines on arbitrary $\mathbb{C}$-algebras, we will also need an analogue or generalization of the dimension of a vector space. This analogue is the rank of a module. To gain some intuition for this, we will define the rank of a projective module over an integral domain (this is justified since irreducible varieties correspond to finitely generated algebras with no zero divisors).

Definition 2.2. Let R be an integral domain, P be a projective R-module and F the field of fractions of R. Then the rank of P is

$$\operatorname{rank} P = \dim_F P.$$

We define the *functor of lines*

$$G_{1,n} : \mathbf{AffAlg}_{\mathbb{C}} \to \mathbf{Sets}$$

via

$$\begin{aligned} G_{1,n}(R) &= \{M \leq R^n \mid R^n/M \text{ projective of constant rank } n-1\} \\ &= \{M \leq R^n \mid R^n \cong M \oplus N \text{ for some } N \text{ and rank } M = 1\} \end{aligned}$$

That is, instead of subspace we want a submodule and instead 1-dimensional, we want quotient of rank $n-1$.

Proposition 2.3. *We have an isomorphism of functors $h_{\mathbb{P}^{n-1}} \cong G_{1,n}$.*

That is, for a fixed K-algebra R, we have a map

$$\Phi_R : G_{1,n}(R) \to h_{\mathbb{P}^{n-1}}(R)$$

which to each submodule $M \leq R^n$ with R^n/M projective of constant rank $n-1$ associates $\Phi_R(M) \in \operatorname{Hom}(R, \mathbb{P}^{n-1})$ such that the following diagram commutes:

$$\begin{array}{ccc} G_{1,n}(R) & \xrightarrow{\Phi_R} & h_{\mathbb{P}^{n-1}}(R) \\ {\scriptstyle G_{1,n}(f)}\downarrow & & \downarrow{\scriptstyle h_{\mathbb{P}^{n-1}}(f)} \\ G_{1,n}(S) & \xrightarrow{\Phi_S} & h_{\mathbb{P}^{n-1}}(S) \end{array}$$

We also have a map

$$\Psi_R : h_{\mathbb{P}^{n-1}}(R) \to G_V(R)$$

which to each $f \in \operatorname{Hom}(R, \mathbb{P}^{n-1})$ associates a submodule $\Psi_R(f) \leq R^n$ such that $R^n/\Psi_R(f)$ is projective of constant rank $n-1$ and such that the following diagram commutes:

$$\begin{array}{ccc} G_{1,n}(R) & \xleftarrow{\Psi_R} & h_{\mathbb{P}^{n-1}}(R) \\ {\scriptstyle G_{1,n}(f)}\downarrow & & \downarrow{\scriptstyle h_{\mathbb{P}^{n-1}}(f)} \\ G_{1,n}(S) & \xleftarrow{\Psi_S} & h_{\mathbb{P}^{n-1}}(S) \end{array}$$

That is, we have natural transformations Φ and Ψ and we must have that they are mutually inverse. That is $\Phi \circ \Psi \cong \operatorname{Id}$ and $\Psi \circ \Phi \cong \operatorname{Id}$. To show this association explicitly would make use of more advanced machinery and so we leave this unproven for now.

Notice that $G_{1,n}(\mathbb{C}) \cong \mathbb{P}^{n-1}$.

Thus, the functor of lines is a representable functor, with representing object $\mathbb{P}^n$. So we see that $\mathbb{P}^n$ is the moduli space of lines in K^{n+1}, and the above discussion gives us some insight into how these lines vary in families.

3 The Subspace Functor

We will first define the subspace functor analogously to the functor of lines. That is, for a given $\mathbb{C}$-algebra R, rather than looking at submodules M of R^n for which R^n/M is projective of constant rank $n-1$ (that is, instead of *lines* in R^n), we can consider modules M for which R^n/M is projective of constant rank $n-k$ for $k \leq n$ (*subspaces* of R^n). This is analogous to looking at k-dimensional subspaces in $\mathbb{C}^n$ as we first saw our Grassmannian defined. We define the *subspace functor*

$$G_{k,n} : \mathbf{AffAlg}_{\mathbb{C}} \to \mathbf{Sets}$$

via

$$\begin{aligned} G_{k,n}(R) &= \{M \leq R^n \mid R^n/M \,\text{projective of rank}\, n-k\} \\ &= \{M \leq R^n \mid R^n \cong M \oplus N \,\text{for some}\, N \,\text{and rank}\, M = k\}. \end{aligned}$$

Notice that $G_{k,n}(\mathbb{C})) \cong Gr(k,n)$. We would like to incorporate our discussion of the Pluücker embedding into our current analysis. Let I_{Pl} be the ideal generated by the Plücker equations, and let $V = Z(I_{Pl})$. Then we have the following result:

Proposition 3.1. *There is an isomorphism of functors $h_V \cong G_{k,n}$.*

That is, the subspace functor is representable and is represented by the variety cut out by the Plücker equations, which is precisely $Gr(k,n)$.

4 A Few More Words on the Plücker Embedding

We want to make the use of the Plücker embedding more explicit. We will define the Plücker functor by generalizing this embedding and arrive at a functor isomorphic to the subspace functor.

Let n be a positive integer and let $W \subset \mathbb{C}^n$ be a k-dimensional subspace spanned by the vectors $v_1, ..., v_k$. We associate to W the multivector

$$\lambda = v_1 \wedge \cdots \wedge v_k \in \wedge^k \mathbb{C}^n$$

If we choose a different basis, say $w_1, ..., w_k$ for W, and if P is the change of basis matrix, we have

$$\lambda = v_1 \wedge \cdots \wedge v_k = \det(P) w_1 \wedge \cdots \wedge w_k$$

Thus, λ is determined up to scalars. Define

$$Pl : Gr(k,n) \to \mathbb{P}(\wedge^k \mathbb{C}^n)$$

via

$$W \mapsto [\lambda].$$

By the discussion in Harris's *Algebraic Geometry*, $[\lambda]$ is in the image of Grassmannian if and only if λ is *totally decomposable*:

$$\lambda = \alpha w_1 \wedge \cdots \wedge w_k.$$

We want to characterize the subspace functor using this idea. Let $F_{k,n}$ be the *Plücker functor*, defined by

$$F_{k,n}(R) = \{M \leq \wedge^k R^n \mid \wedge^k R^n / M \,\text{projective of constant rank}\, n-1 \text{ and}$$

$$\forall \mathfrak{p} \in \operatorname{Spec} R, M_{\mathfrak{p}} = R_{\mathfrak{p}} w_1 \wedge \cdots \wedge w_k\}.$$

Proposition 4.1. *There is an isomorphism of functors $F_{k,n} \cong G_{k,n}$.*