Chow Groups with Coefficients and Generalized Severi-Brauer Varieties

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Introduction

The study of cycles on Severi-Brauer varieties provides a beautiful blend of algebra and geometry. While interesting in its own right, the study of K-cohomology and algebraic cycles has seen wide-ranging applications.

- Main objects of study: Central simple algebras and involutions via associated Severi-Brauer and involution varieties.
- Main tools: Algebraic K-theory and K-cohomology.
- Current work on K_1-zero-cycles for some generalized Severi-Brauer varieties.

Throughout, F will denote an arbitrary field whose characteristic is not 2.

Central Simple Algebras

Let A be a central simple algebra over F. Recall that we may write $A \cong M_n(D)$, for a division algebra D which is unique up to isomorphism.

- Define the *degree* of A to be $\sqrt{\dim_F(A)}$.
- Define the *index* of A to be $\deg(D)$, the degree of its underlying division algebra.
- A central simple algebra A is a division algebra if and only if $\deg(A) = \mathrm{ind}(A)$.
- If $L \otimes_F A \cong M_n(L)$, we say that L is a *splitting field* for A, or that L splits A.
- The collection of central simple algebras of degree n is parametrized by the cohomology set $H^1(F, \mathrm{PGL}_n)$.

Involutions

- An *algebra with involution* is a pair (A, σ) consisting of a central simple algebra A and an anti-automorphism $\sigma : A \to A$ satisfying $\sigma^2 = \mathrm{id}_A$.
- An involution σ *of the first kind* satisfies $\sigma|_F = \mathrm{id}_F$. If σ is a twisted form of a symmetric bilinear form, we say that σ is *orthogonal*. Otherwise, it is *symplectic*.
- An involution which induces a degree 2 automorphism of F is *of the second kind* or *unitary*.
- We extend the notion of "algebras with unitary involution" to include semi-simple F-algebras of the form $A_1 \times A_2$, where each A_i is central simple over F. The center L of such an algebra is either $F \times F$ or a separable quadratic extension of F. For example, $(A \times A^{\mathrm{op}}, \varepsilon)$, where ε is the involution which exchanges factors.

To any central simple algebra A of degree n, we may associate the variety

$$\mathrm{SB}(A) = \{I \lhd_r A \,|\, \dim_F I = n\} \subset \mathrm{Gr}(n, n^2)$$

of right ideals of A of dimension $\deg(A)$.

- The variety $\mathrm{SB}(A)$ is a *Severi-Brauer variety* over F, or a *twisted form* of projective space, i.e., $\mathrm{SB}(A)_{F_{\mathrm{sep}}} \cong \mathbb{P}^{n-1}_{F_{\mathrm{sep}}}$.
- The variety $\mathrm{SB}(A)$ has a rational point over L/F if and only if L splits A.
- The collection of Severi-Brauer varieties of dimension $n-1$ is parametrized by the set $H^1(F, \mathrm{PGL}_n)$.

Examples

- $\mathrm{SB}(M_n(F)) \cong \mathbb{P}^{n-1}$
- Let $A = (a, b)_F$. Then $\mathrm{SB}(A) = \{ax^2 + by^2 - z^2 = 0\} \subset \mathbb{P}^2$.

More generally, for any integer $1 \leq k \leq n$, let

$$\mathrm{SB}_k(A) = \{I \lhd_r A \mid \dim_F I = nk\} \subset \mathrm{Gr}(nk, n^2)$$

denote the collection of right ideals of dimension nk.

- The variety $\mathrm{SB}_k(A)$ is a twisted form of the Grassmannian $\mathrm{Gr}(k, n)$.
- $\mathrm{SB}_1(A) = \mathrm{SB}(A)$.
- The variety $\mathrm{SB}_k(A)$ has a rational point over L/F if and only if $\mathrm{ind}(A_L) \mid k$.
- The elements of $\mathrm{SB}_k(A)$ are ideals of reduced dimension k, where reduced dimension is

$$\mathrm{rdim}(I) = \frac{\dim(I)}{\deg(A)}.$$

Involution Varieties

Let (A, σ) be a central simple algebra of degree n with orthogonal involution. To this pair we associate the variety of *isotropic ideals*

$$\mathrm{IV}(A, \sigma) = \{I \triangleleft_r A \mid \dim_F(I) = n,\ \sigma(I) \cdot I = 0\}.$$

- There is an inclusion $\mathrm{IV}(A, \sigma) \subset \mathrm{SB}(A)$.
- The variety $\mathrm{IV}(A, \sigma)$ is a twisted projective quadric, i.e., after sufficiently extending scalars, σ is the adjoint involution of a quadratic form q_σ. Then

$$\mathrm{IV}(A, \sigma) \cong V(q_\sigma) \subset \mathbb{P}^{n-1} \cong \mathrm{SB}(A).$$

Some Algebraic Groups

$\mathrm{GL}(A) = A^{\times}$. The kernel of the reduced norm homomorphism $\mathrm{Nrd}_A : \mathrm{GL}(A) \to F^{\times}$ is $\mathrm{SL}(A)$.

For (A, σ) an algebra of degree $n = 2m$ with unitary involution:

- $\mathrm{GU}(A, \sigma) = \{x \in A \mid \mu(x) := \sigma(x)x \in F^{\times}\}$
- $\mathrm{SGU}(A, \sigma) = \{x \in \mathrm{GU}(A, \sigma) \mid \mathrm{Nrd}_A(x) = \mu(x)^m\}$
- $\mathrm{SU}(A, \sigma) = \{x \in \mathrm{GU}(A, \sigma) \mid \mathrm{Nrd}_A(x) = 1\}$

For (A, σ) an algebra with orthogonal involution, there are groups $\Gamma(A, \sigma)$ and $\mathrm{Spin}(A, \sigma)$, called the *Clifford* and *spin* groups of (A, σ).

In the split case, $A = \mathrm{End}_F(V)$ and σ corresponds to a quadratic form on V, $\Gamma(A, \sigma)$ is the subgroup of the multiplicative group of the even Clifford algebra $C_0(V, q)$ consisting of elements whose conjugation action fixes $V \subset C_0(V, q)$. $\mathrm{Spin}(A, \sigma)$ is the kernel of the *spinor norm*.

Chow Theory

The theory of Chow groups with coefficients, as developed by M. Rost, defines coefficient systems for the analogue of singular homology for varieties (i.e., Chow groups of cycles of a given dimension).

$$\begin{array}{ccc} \text{CW complex } X & \longleftrightarrow & \text{Smooth variety } X \\ H_p(X, \mathbb{Z}) & \longleftrightarrow & A_p(X, K_n) \\ H_p(X, R) & \longleftrightarrow & A_p(X, M) \end{array}$$

The Chow group $\mathrm{CH}_p(X)$ of p-dimensional cycles on a variety X is the cokernel of the divisor map

$$\bigoplus_{y \in X_{(p+1)}} F(y)^\times \longrightarrow \bigoplus_{x \in X_{(p)}} \mathbb{Z}.$$

Of course, $F(y)^\times = K_1(F(y))$ and $\mathbb{Z} = K_0(F(x))$. We wish to generalize this situation to higher K-groups.

Milnor K-Theory of Fields

Definition

The n^{th} Milnor K-group of F is the quotient

$$K_n(F) = T^n(F^*)/\langle a \otimes (1-a) \mid a \in F^* \rangle.$$

The Milnor K-theory ring of F is the direct sum of the $K_n^M(F)$, i.e.,

$$K_*(F) = T(F^*)/\langle a \otimes (1-a) \mid a \in F^* \rangle.$$

- We denote the image of an element $a_1 \otimes \cdots \otimes a_n$ in $K_n(F)$ by $\{a_1, \ldots, a_n\}$.
- Quillen and Milnor K-theory (of fields) coincide for $n = 0, 1, 2$.

Residue Homomorphism

Let L be a field with discrete valuation v and residue field κ. The valuation $v : L^* \to \mathbb{Z}$ can be viewed as a homomorphism $K_1(L) \to K_0(\kappa)$. In fact, for every $n \geq 0$, there is a residue homomorphism

$$\partial_v : K_{n+1}(L) \to K_n(\kappa)$$

determined by the following condition: if $a_0, ..., a_n \in L^*$ with $v(a_i) = 0$ for all $i = 1, ..., n$ then

$$\partial_v(\{a_0, ..., a_n\}) = v(a_0)\{\bar{a}_1, ..., \bar{a}_n\}.$$

The residue homomorphism $K_2(L) \to K_1(\kappa)$ differs from the *tame symbol* by a sign.

For a smooth variety X, the residue maps induce a homomorphism

$$\partial : \bigoplus_{y \in X_{(p)}} K_n(F(y)) \longrightarrow \bigoplus_{x \in X_{(p-1)}} K_{n-1}(F(x)).$$

- Let $y \in X_{(p)}$ and let $x \in X_{(p-1)}$.
- Let Y be the closure of $\{y\}$ in X, considered as a subvariety of X of dimension p.
- If $x \in Y$, then $\mathcal{O}_{Y,x}$ is a DVR, with valuation v_x.
- Set the (x, y)-component ∂_x^y of ∂ to be ∂_{v_x}.
- Otherwise, set $\partial_x^y = 0$

We then define $A_p(X, K_n)$ to be the homology at the middle term of

$$\cdots \to \bigoplus_{z \in X_{(p+1)}} K_{n+1}(F(z)) \xrightarrow{\partial} \bigoplus_{y \in X_{(p)}} K_n(F(y)) \xrightarrow{\partial} \bigoplus_{x \in X_{(p-1)}} K_{n-1}(F(x)) \to \cdots$$

- The diagonal terms have a nice description:

$$A_p(X, K_p) \cong \mathrm{CH}_p(X).$$

- For the remainder, we will mostly be interested in the group of K_1-zero-cycles:

$$A_0(X, K_1) = \mathrm{coker}\left(\bigoplus_{y \in X_{(1)}} K_2(F(y)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} K_1(F(x))\right)$$

- A K_1-zero-cycle on a variety X is thus an equivalence class of formal sums $\sum(\{\alpha\}, x)$, where $x \in X$ is a closed point and $\alpha \in F(x)^\times = K_1(F(x))$.

What about sheaf cohomology?

- Quillen's algebraic K-theory is defined as a sequence of functors

$$K_n^Q : \underline{\mathrm{Sch}}_F \xrightarrow{\mathcal{P}} \underline{\mathrm{Ex}} \xrightarrow{Q} \underline{\mathrm{Cat}} \xrightarrow{B} \underline{\mathrm{Top}} \xrightarrow{\pi_{n+1}} \underline{\mathrm{Ab}}.$$

- Zariski sheafification of $U \mapsto K_n^Q(U)$, for any open subscheme $U \subset X$, allows one to define cohomology groups $H^p(X, K_n^Q)$.
- The agreement of Quillen and Milnor K-theory of fields in degrees 0, 1, 2 implies $H_p(X, K_n^Q) \cong A_p(X, K_n)$ for $n = 0, 1, 2$.

R-Equivalence

- Let G be an algebraic group over F.
- A point $x \in G(F)$ is called R-trivial if there is a rational morphism $f : \mathbb{P}^1 \dashrightarrow G$, defined at 0 and 1, and with $f(0) = 1$ and $f(1) = x$.
- The collection of all R-trivial elements of $G(F)$ is denoted $RG(F)$ and is a normal subgroup of $G(F)$.
- If H is a normal closed subgroup of G then $RH(F)$ is a normal subgroup of $G(F)$

The notion of R-equivalnce has played an important role in computing K-groups and K-cohomology groups, exhibited by the following theorem of Voskresenskiĭ.

- The abelianization map $\mathrm{GL}(A) = A^\times \to A^\times_{ab} = K_1(A)$ induces an isomorphism

$$\mathrm{GL}(A)/R\mathrm{SL}(A) \cong K_1(A).$$

Some Results on K-Cohomology Groups

- Panin ('84): Zero-cycles on Severi-Brauer varieties.
- Merkurjev-Suslin ('92): $A_0(\mathrm{SB}(A), K_1) \xrightarrow{\sim} K_1(A)$.
- Merkurjev ('95): $A^1(X, K_2)$ for projective homogeneous varieties.
- Krashen ('00): Zero-cycles on Severi-Brauer flag varieties, (generalized) involution varieties.
- Chernousov-Merkurjev ('00): Zero-cycles on generalized Severi-Brauer, involution varieties, homogeneous varieties of exceptional types.
- Chernousov-Merkurjev ('01): $A_0(X, K_1) \cong \Gamma(A,\sigma)/R\mathrm{Spin}(A,\sigma)$ for $X = \mathrm{IV}(A,\sigma)$.

Theorem (Rost-Voevodsky '09, Bloch-Kato Conjecture)

Let F be a field and ℓ invertible in F. Then the norm-residue homomorphism

$$K_n^M(F)/\ell \longrightarrow H^n(F, \mu_\ell^{\otimes 2})$$

is an isomorphism.

- The case $n = 2$ is the Merkurjev-Suslin Theorem.
- The case $\ell = 2$ is the Milnor Conjecture.

Theorem (Merkurjev '15, Suslin's Conjecture)

Let A be a central simple F-algebra. If $\operatorname{ind}(A)$ *is not square-free, then there is a field extension L/F such that*

$$\mathrm{SK}_1(A \otimes_F L) \neq 0$$

Outline

Our main goal is to compute the group of K_1-zero-cycles of the 2^{nd} generalized Severi-Brauer variety of an algebra A of index 4. We will complete this in the following steps:

- For $A = M_n(D)$, relate the group of K_1-zero-cycles on $\mathrm{SB}_2(A)$ to that of $\mathrm{SB}_2(D)$.
- A result of Krashen yields an isomorphism $\mathrm{SB}_2(D) \cong \mathrm{IV}(B, \sigma)$ for degree 6 algebra B of index ≤ 2 with orthogonal involution σ.
- A result of Chernousov-Merkurjev computes the group $A_0(\mathrm{IV}(B, \sigma), K_1)$ in terms of the Clifford group $\Gamma(B, \sigma)$ and spin group $\mathrm{Spin}(B, \sigma)$.
- Certain exceptional identifications allows one to compute the Clifford and spin groups of (B, σ) entirely in terms of unitary groups of the algebra with unitary involution $(D \times D^{\mathrm{op}}, \varepsilon)$.

Reduction to Algebras of Square Degree

Proposition (M)

Let $A = M_n(D)$ *be a central simple algebra of index* 4, $X = \mathrm{SB}_2(A)$ *and* $Y = \mathrm{SB}_2(D)$. *There is an isomorphism*

$$A_0(X, K_1) \cong A_0(Y, K_1).$$

- Since A has index 4, the variety $\mathrm{SB}_4(A)$ has an F-rational point, so there is an ideal $J \lhd_r A$ of reduced dimension 4.
- Let $A = \mathrm{End}_D(V)$. Then $J = \mathrm{Hom}(V, W)$ for some 4-dimensional D-subspace $W \subset V$. Take e_J to be the idempotent of A given by projection onto W.
- Define a rational map $\varphi_J : \mathrm{SB}_2(A) \dashrightarrow \mathrm{SB}_2(J)$ via $I \mapsto e_J I \subset J$. Notice that this map is well-defined as long as $\mathrm{rdim}(e_J I) = 2$.

Reduction to Algebras of Square Degree

- The underlying division algebra D of A satisfies $\mathrm{SB}_2(J) \cong \mathrm{SB}_2(D)$, yielding a map $\mathrm{SB}_2(A) \to \mathrm{SB}_2(D)$. [Krashen]
- The general fiber $\mathfrak{f}$ is a rational variety. [Blanchet]
- $A_0(\mathrm{SB}_2(D) \times \mathfrak{f}, K_1) \cong A_0(\mathrm{SB}_2(D), K_1)$ [Krashen]
- The groups $A_0(X, K_n)$ are birational invariants of smooth proper varieties X [Rost-Merkurjev], so that

$$A_0(\mathrm{SB}_2(A), K_1) \cong A_0(\mathrm{SB}_2(D) \times \mathfrak{f}, K_1).$$

Connection to Algebraic Groups

Theorem (Krashen)

Let D be a degree 4 central simple algebra over F. Then $\mathrm{SB}_2(D)$ is isomorphic to an involution variety $\mathrm{IV}(B, \sigma)$ of a degree 6 algebra with orthogonal involution σ. Moreover, $\mathrm{ind}(B) \leq 2$.

Theorem (Chernousov-Merkurjev)

Let B be a central simple F-algebra of even dimension, $\mathrm{ind}(B) \leq 2$ *with orthogonal involution σ. Let X be the corresponding involution variety. Then there is a canonical isomorphism*

$$\Gamma(B, \sigma)/R\,\mathrm{Spin}(B, \sigma) \cong A_0(X, K_1).$$

- Exceptional identifications allow us to compute the Clifford and spin groups in terms of unitary groups of $(D \times D^{\mathrm{op}}, \varepsilon)$.

$$\Gamma(B, \sigma) \cong \mathrm{SGU}(D \times D^{\mathrm{op}}, \varepsilon) = \{(x, \alpha) \in D^{\times} \times F^{\times} \mid \mathrm{Nrd}_D(x) = \alpha^2\}$$

$$\mathrm{Spin}(B, \sigma) \cong \mathrm{SL}(D)$$

- We conclude that the quotient $\Gamma(B, \sigma)/R\mathrm{Spin}(B, \sigma)$ is given by

$$\{(x, \alpha) \in D^{\times} \times F^{\times} \mid \mathrm{Nrd}_D(x) = \alpha^2\}/R\mathrm{SL}(D).$$

- Lastly, we use $K_1(D) = D^{\times}/R\mathrm{SL}(D)$.

Main Result

Theorem (M)

Let A be a central simple algebra of index 4 and arbitrary degree over a field F, and let $X = \mathrm{SB}_2(A)$. The group $A_0(X, K_1)$ can be identified as the subgroup of $K_1(A)$ consisting of elements whose reduced norm is a square in F.

Compare to the result of Merkurjev-Suslin on K_1-zero-cycles of Severi-Brauer varieties:

Theorem (Merkurjev-Suslin)

Let A be a central simple algebra over a field F and $X = \mathrm{SB}(A)$. There is an isomorphism

$$A_0(X, K_1) \cong K_1(A).$$

Future Work

- The reduction to (division) algebras of degree 4 generalizes to algebras of degree p^2. It then seems reasonable to expect a generalization of the main result to algebras of index p^2, possibly using more exotic cycle modules and Galois-theoretic techniques.
- Does the reduction to algebras of degree p^2 come from a motivic decomposition of $\mathrm{SB}_2(A)$ in terms of $\mathrm{SB}_2(D)$? (Brosnan)
- The Brown-Gersten-Quillen spectral sequence computes K-groups of schemes in terms of K-cohomology groups. Can our main result along with the known computation of the K-groups of generalized Severi-Brauer varieties (LSW) be utilized to give descriptions of other K-cohomology groups of $\mathrm{SB}_2(A)$?
- There is a spectral sequence computing K-theory from motivic cohomology groups. Since $A_0(X, K_1)$ arises as the $(-1,-1)$ motivic cohomology group, can we compute other motivic cohomology groups of $\mathrm{SB}_2(A)$?

Thank you.

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