K-Theory of Algebraic Curves

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Given a scheme X, there are two important techniques one may use to study X. On one hand, there is the theory which utilizes subschemes or divisors and algebraic cycles and its corresponding cohomology theory is represented by the Chow ring $A(X)$. On the other hand, there is the method of studying bundles over X, utilizing vector bundles and coherent and quasicoherent sheaves, and its corresponding cohomology theory is K-theory. This is analogous to the methods used in differential geometry relating submanifolds and vector bundles.

Our main goal will be to completely characterize the Grothendieck group of a nonsingular algebraic curve in terms of its Picard group. We begin with a few definitions.

Definition 1.1. Let X be a noetherian scheme and let $\mathcal{C}$ be the category of coherent sheaves on X. Let $\mathbb{Z}[\mathcal{C}]$ be the free abelian group generated by isomorphism classes $[\mathscr{F}]$ where $\mathscr{F} \in \operatorname{ob}\mathcal{C}$. For each exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

of coherent sheaves, form the element $[\mathscr{F}_2]-[\mathscr{F}_1]-[\mathscr{F}_3]$ and let $\mathcal{J}$ be the subgroup generated by such elements. Define

$$K_\cdot(X) = \mathbb{Z}[\mathcal{C}]/\mathcal{J}.$$

Remark 1.2. The group $K_\cdot(X)$ defined above is often denoted $G_0(X)$. One may similarly define $K_0(X)$ or $K^\cdot(X)$ using locally free sheaves: Let $\mathcal{L}$ be the category of locally free sheaves on X, and let $\mathcal{J}'$ be the subgroup of $\mathbb{Z}[\mathcal{L}]$ generated by elements of the form $[\mathscr{F}_2]-[\mathscr{F}_1]-[\mathscr{F}_3]$ as above. Define

$$K_0(X) = K^\cdot(X) = \mathbb{Z}[\mathcal{C}]/\mathcal{J}'.$$

In certain cases, these groups are all isomorphic, as we will see shortly. We first need some preliminary results.

Lemma 1.3. *Let X be a noetherian, integral, separated, regular scheme, such as a nonsingular curve. Then any coherent sheaf $\mathscr{F}$ on X has a finite locally free resolution.*

We will prove this lemma assuming a few important facts:

Fact 1: (HAG Ex. III.6.8) Let X be a noetherian scheme. Then $\mathrm{Coh}(X)$, the category of coherent sheaves on X, has enough locally frees. That is, every coherent sheaf on X is a quotient of a locally free sheaf. Thus, every coherent sheaf admits a locally free resolution

$$\mathscr{E}. \to \mathscr{F}$$

Fact 2: (HAG Prop. III.6.11 A) Let A be a regular local ring and M an A-module. Then

$$\operatorname{pdim} M \leq \dim A$$

where $\operatorname{pdim} M$ is the minimum length of a projective resolution of M.
Fact 3: (HAG Ex. III.6.5 C) For a coherent sheaf $\mathscr{F}$ on X,

$$\operatorname{hdim} \mathscr{F} = \sup_{x \in X} \operatorname{pdim}_{\mathcal{O}_X} \mathscr{F}_x$$

where $\operatorname{hdim} \mathscr{F}$ is the minimum length of a locally free resolution of $\mathscr{F}$.

Proof. (Lemma 1.3) Let $\mathscr{F}$ be a coherent sheaf on X and let $x \in X$. By Fact 1, $\mathscr{F}$ admits a locally free resolution, so $\operatorname{hdim} \mathscr{F}$ exists. Since X is regular, $\mathcal{O}_{X,x}$ is a regular local ring and thus by Fact 2 we have

$$\operatorname{pdim} \mathscr{F}_x \leq \dim \mathcal{O}_{X,x}.$$

Since x was arbitrary, using Fact 3 we have

$$\operatorname{hdim} \mathscr{F} = \sup_{x \in X} \operatorname{pdim} \mathscr{F}_x \leq \dim \mathcal{O}_{X,x} < \infty.$$

Thus, $\mathscr{F}$ admits a finite locally free resolution. □

Theorem 1.4. *If X is a noetherian, integral, separated, regular scheme, then the natural map*

$$i : K^{\cdot}(X) \to K.(X)$$

or

$$i : K_0(X) \to G_0(X)$$

is an isomorphism.

Proof. Let $[\mathscr{F}] \in K.(X)$. By Lemma 1.3, $\mathscr{F}$ has a finite resolution by locally free sheaves

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

Then

$$[\mathscr{F}] = \sum_{i=0}^{n} (-1)^n [\mathscr{E}_i]$$

in $K.(X)$. Thus we can take $\sum_{i=0}^{n} (-1)^n [\mathscr{E}_i]$ as a preimage for $[\mathscr{F}]$.
To see injectivity, we define a candidate for the inverse of i. Let $\mathscr{F}$ be a coherent sheaf and let $\mathscr{E}. \to \mathscr{F}$ be its finite locally free resolution. Define $j([\mathscr{F}]) = \sum (-1)^i [\mathscr{E}_i]$. One checks that j is independent of the choice of resolution of $\mathscr{F}$ (we will check this below), so that j is a well-defined group homomorphism. From the definitions of i and j we have $j \circ i = \mathrm{id}$ so that i is injective. Thus, i is an isomorphism. □

Remark 1.5. In this case, we use $K_0(X)$ to denote the common value of $K^{\cdot}(X), K_{\cdot}(X), G_0(X)$ and $K_0(X)$. If C is a nonsingular curve, then of course C is noetherian, integral, separated and regular. Thus,

$$K_0(C) = K^{\cdot}(C) \cong K_{\cdot}(C) = G_0(C).$$

One may have first encountered the K-theory of a ring. That is, $K_0(A)$ is the group completion of the commutative monoid of isomorphism classes of finitely generated projective modules over A under direct sum. Our definition coincides with this one, so that we may think of the K-theory of schemes as a generalization of the K-theory of rings.

Proposition 1.6. *Let A be a noetherian ring and let $X = \operatorname{Spec} A$. Then we have a group isomorphism $K_0(A) \cong K_0(X)$.*

Proof. By HAG Corollary 5.5, we have an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_X$-modules. Furthermore, if P is a projective A-module, then $\widetilde{P}$ is locally free. If $\mathscr{F}$ is locally free, it can be made trivial on open sets of the form $U_i = D(s_i)$ so that $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for free modules M_i. The isomorphisms on the overlaps give open patching data defining a projective A-module P (see Weibel I.2.5 for details). □

Example 1.7. $K_0(\mathbb{A}^1)$. Let $\mathscr{F}$ be a coherent sheaf on X. Then $\mathscr{F}$ is locally $\widetilde{M}$ for some finitely generated $k[x]$-module M. Giving M a presentation by generators and relations, we have

$$k[t]^{\oplus n} \to k[t]^{\oplus m} \to M \to 0.$$

We can also assume the first homomorphism is injective. For if there were nontrivial kernel, since $k[t]$ is an integral PID, any submodule of $k[t]^{\oplus n}$ is free. Thus, we can reduce n appropriately. We then have an exact sequence

$$0 \to k[t]^{\oplus n} \to k[t]^{\oplus m} \to M \to 0.$$

Applying $\widetilde{(-)}$, which is exact by HAG Proposition 5.4A, we obtain the short exact sequence

$$0 \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m} \to \mathscr{F} \to 0$$

and thus $[\mathscr{F}] = (m-n)[\mathcal{O}_X]$ in $K_0(X)$. Since $\mathscr{F}$ was arbitrary, it follows that the map

$$\varphi : \mathbb{Z} \to K_0(X)$$

given by $n \mapsto n[\mathcal{O}_X]$ is surjective.

We will now define the rank function rank : $K_0(X) \to \mathbb{Z}$ by

$$\operatorname{rank}(\mathscr{F}) = \dim_K \mathscr{F}_\eta$$

where η is the generic point of X and $K = \mathcal{O}_\eta$ is the function field of X. We want to see that rank is a well-defined homomorphism: Given a short exact sequence of coherent sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

and passing to the stalks at the generic point, we obtain a short exact sequence of finite dimensional K-vector spaces

$$0 \to \mathscr{F}'_\eta \to \mathscr{F}_\eta \to \mathscr{F}''_\eta \to 0.$$

Thus, $\operatorname{rank} \mathscr{F} = \operatorname{rank} \mathscr{F}' + \operatorname{rank} \mathscr{F}''$, so rank is a well-defined homomorphism. To see that it is surjective, notice that $\operatorname{rank} \mathcal{O}_X = 1$ so that for any $n \in \mathbb{Z}$, we have $\operatorname{rank} \mathcal{O}_X^{\oplus n} = n$. Thus, our above homomorphism φ splits and so $K_0(\mathbb{A}^1) \cong \mathbb{Z}$.

Alternatively, using our proposition above, we can compute $K_0(X)$ much more easily. Since $\mathbb{A}^1$ is affine, with affine coordinate ring $k[t]$, we have

$$K_0(X) \cong K_0(k[t]).$$

Since $k[t]$ is a PID, any projective $k[t]$-module must be free and thus we have an isomorphism $K_0(k[t]) \cong \mathbb{Z}$ given by $k[t]^{\oplus n} \mapsto n$.

We have now seen that for a nonsingular curve, the information given to us by locally free sheaves and is the same as that given to us by coherent sheaves (in fact, this is true for any noetherian regular scheme with an ample line bundle). We can do even better. For a nonsingular curve, the information obtained from coherent sheaves can be just as well obtained from invertible sheaves.

Proposition 1.8. *Let C be a nonsingular curve. If $\mathscr{F}$ is any coherent sheaf of rank r on C, there is a divisor D on X and an exact sequence*

$$0 \to \mathcal{O}_C(D) \to \mathscr{F} \to \mathscr{T} \to 0,$$

where $\mathscr{T}$ is a torsion sheaf.

Proof. Let $\mathscr{F}$ be a coherent sheaf of rank r and let $\mathscr{L}$ be an ample invertible sheaf on C (such a sheaf exists since C is necessarily projective). Then there exists $n > 0$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. Let $s_1, ..., s_m$ generate $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$. This gives a surjective map

$$\mathcal{O}_C^{\oplus m} \twoheadrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes n}.$$

Let η be the generic point of X. We then have a map of $K(C)$-vector spaces

$$(\mathcal{O}_{C,\eta})^{\oplus m} \twoheadrightarrow \mathscr{F}_\eta,$$

which gives

$$K(C)^m \twoheadrightarrow \mathscr{F}_\eta.$$

Thus, there is an $r > 0$ such that $K(C)^r \cong \mathscr{F}_\eta$. Thus, $\mathcal{O}_C^{\oplus r}$ is generically isomorphic to $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$. That is, there is a dense open $U \subset C$ such that

$$\mathcal{O}_U^{\oplus r} \cong_\varphi \mathscr{F} \otimes \mathscr{L}^{\otimes n}|_U.$$

Thus, $\ker\varphi$ is a skyscraper sheaf at the finitely many points of $C \setminus U$. We have an exact sequence

$$0 \to \ker\varphi \to \mathcal{O}_C^{\oplus r} \to \mathscr{F} \otimes \mathscr{L}^{\otimes n} \to \operatorname{coker}\varphi \to 0.$$

Let $p \in C \setminus U$. Passing to the stalks, we have

$$0 \to (\ker\varphi)_p \to (\mathcal{O}_C^{\oplus r})_p \to (\mathscr{F} \otimes \mathscr{L}^{\otimes n})_p \to \cdots$$

Now, $(\ker\varphi)_p$ is a submodule of a free module (so torsion free) over $\mathcal{O}_{X,p}$, a PID and is thus a free module. But $\ker\varphi$ is a skyscraper at p and so we must have $\ker\varphi = 0$. Thus, our exact sequence becomes

$$0 \to \mathcal{O}_C^{\oplus r} \to \mathscr{F} \otimes \mathscr{L}^{\otimes n} \to \operatorname{coker}\varphi \to 0.$$

Since $\mathscr{L}^{\otimes n}$ is a line bundle, so is its inverse, which is thus isomorphic to $\mathcal{O}_C(D)$ for some divisor D. Tensoring the above exact sequence with this $\mathcal{O}_C(D)$ we obtain

$$0 \to (\mathcal{O}_C(D))^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0,$$

where $\mathscr{T}$ is $\operatorname{coker}((\mathcal{O}_C(D))^{\oplus r} \to \mathscr{F})$. We want to see that $\mathscr{T}$ is torsion. Consider the induced short exact sequence on the stalks at the generic point η:

$$0 \to (\mathcal{O}_C(D))_\eta^{\oplus r} \to \mathscr{F}_\eta \to \mathscr{T}_\eta \to 0$$

This is a short exact sequence of finite dimensional $K(C)$-vector spaces. Notice that $\dim_{K(X)}(\mathcal{O}_C(D))_\eta^{\oplus r} = \dim_{K(X)} \mathscr{F}_\eta$. Thus $\mathscr{T}_\eta = 0$ so $\mathscr{T}$ is torsion. □

We now come to our final result which makes our above discussion even more precise.

Proposition 1.9. *Let C be a nonsingular curve. Then*

$$K_0(C) \cong \operatorname{Pic}(C) \oplus \mathbb{Z}.$$

Proof. We begin by defining the following maps:
Determinant: By our discussion above, any coherent sheaf $\mathscr{F}$ on C has a finite locally free resolution of length at most 1. That is, there exist locally free sheaves $\mathscr{E}_0, \mathscr{E}_1$ such that we have a short exact sequence

$$0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

Let $r_i = \operatorname{rank}\mathscr{E}_i$ and define $\det\mathscr{F} = (\bigwedge^{r_0}\mathscr{E}_0) \otimes (\bigwedge^{r_1}\mathscr{E}_1)^{-1} \in \operatorname{Pic}(C)$. One checks that this gives a homomorphism

$$\det : K_0(C) \to \operatorname{Pic}(C).$$

We have an isomorphism $\operatorname{Cl}(C) \cong \operatorname{Pic}(C)$. Thus, for any element $\mathscr{L} \in \operatorname{Pic}(C)$, we consider its corresponding divisor $D = \sum n_i p_i$. Define $\psi(D) = \sum n_i [k(p_i)] \in K_0(C)$ where $k(p_i)$ is the skyscraper k at the point p_i. One checks that ψ defines a group homomorphism

$$\psi : \operatorname{Pic}(C) \to K_0(C).$$

Using these two maps as well the rank map and its splitting φ defined above, we have the following diagram:

$$\mathrm{Pic}(C) \qquad K_0(C) \qquad \mathbb{Z}$$

One then checks that this gives us a split exact sequence

$$0 \to \mathrm{Pic}(C) \to K_0(C) \to \mathbb{Z} \to 0,$$

hence

$$K_0(C) \cong \mathrm{Pic}(C) \oplus \mathbb{Z}.$$

□

References

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