MOTIVES: WEEK 3 (CATEGORY OF CORRESPONDENCES)

KELLER VANDEBOGERT

Prerequisites/Reminders

All objects considered will be objects in the category of smooth projective varieties over a field k, denoted $\mathrm{Smproj}(k)$. Recall first that a *cycle* is a formal linear combination of irreducible subvarieties; we use the notation $Z^i(X)$ for X a smooth projective variety to denote the codimension i cycles; that is, a formal linear combination of irreducible codimension i subvarieties. Note that

$$Z(X) = \bigoplus_i Z^i(X)$$

For an adequate equivalence relation $\sim$, recall that the set of all cycles $Z \in Z^i(X)$ with $Z \sim 0$ forms a subgroup denoted $Z^i_\sim(X)$. Define:

$$C^i_\sim(X) := Z^i(X)/Z^i_\sim(X)$$

So that

$$C_\sim(X) = \bigoplus_i C^i_\sim(X)$$

If $\sim$ is to mean rational equivalence, we will use the notation $\mathrm{CH}(X)$ for $C_\sim(X)$ (CH is for Chow, and this is typically called the *Chow Ring*). We will always denote by 1_X the identity element of $C_\sim(X)$; moreover, we will often suppress the adequate equivalence relation and write $C(X)$ when no confusion will occur.

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Definition 0.1. Let $f : X \to Y$ be a morphism of k-varieties with $Z \subset X$ an irreducible subvariety. Define

$$\deg(Z/f(Z)) := \begin{cases} [k(Z) : k(f(Z))] & \text{if } \dim f(Z) = \dim Z \\ 0 & \text{if } \dim f(Z) < \dim Z \end{cases}$$

Then,

$$f_*(Z) := \deg(Z/f(Z))f(Z)$$

Remark 0.2. The above definition of degree in a sense counts the number of preimages or "sheets" sitting above the image $f(Z)$.

Before moving onto an example of a pushforward, the following result will be a sort of black box result for us:

Lemma 0.3.

$$CH(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \cong \mathbb{Q}[X_1, \ldots, X_k]/(X_1^{n_1+1}, \ldots, X_k^{n_k+1})$$

(by convention we will always use rational coefficients)

The above Lemma is particularly useful because the intersection product in the Chow ring becomes literally multiplication of polynomials under the above isomorphism. Note that the variable X_i is to denote the class of a general hyperplane (that is, a codimension 1 irreducible subvariety).

Example 0.4. The variable X in $\mathrm{CH}(\mathbb{P}^1) \cong \mathbb{Q}[X]/(X^2)$ corresponds to the class of a point in $\mathbb{P}^1$. The constant polynomial 1 corresponds to all of $\mathbb{P}^1$.

Example 0.5. Let $v : \mathbb{P}^1 \to \mathbb{P}^3$ denote the twisted cubic $[x : y] \mapsto [x^3 : x^2y : xy^2 : y^3]$. By the above lemma, the pushforward should induce a map

$$v_* : \mathbb{Q}[X]/(x^2) \to \mathbb{Z}[Y]/(Y^4)$$

It suffices to look at the image of our generators 1 and X under this map. As v is an embedding, we have that $\deg(Z/f(Z)) = 1$ for any class Z. Let us first consider $v_*(X)$; as X corresponds to the class of a point, the image will be a point in $\mathbb{P}^3$. In $\mathbb{P}^3$, the class of a point corresponds to Y^3, so we know

$$v_*(X) = C \cdot Y^3$$

where C is some constant to be determined. We can find C by considering the intersection number of any hyperplane in $\mathbb{P}^3$ with a point in $\mathbb{P}^3$, which is easily seen to be 1. Thus we deduce $v_*(X) = Y^3$.

Now, we want to look at $v_*(1)$. Again, 1 corresponds to all of $\mathbb{P}^1$; recall that the image of the twisted cubic in $\mathbb{P}^3$ is a codimension 2 subvariety. By the same reasoning as as above, we deduce that

$$v_*(1) = C \cdot Y^2$$

since Y^2 denotes the class of a codimension 2 subvariety. We again find our constant C by considering the number of points of intersection of a general codimension 2 subvariety with a hyperplane of $\mathbb{P}^3$. By Bëzout's Theorem, this number is 3. Extending by linearity, we see that

$$v_*(a + bX) = 3aY^2 + bY^3$$

Example 0.6. Using the same process as above, we can look at the more general map

$$\mathbb{P}^1 \xrightarrow{\quad v_d \quad} \mathbb{P}^d$$

$$[x : y] \longmapsto [x^d : x^{d-1}y : \cdots : y^d]$$

The pushforward induces a $\mathbb{Q}$-linear morphism

$$v_* : \mathbb{Q}[X]/(X^2) \to \mathbb{Q}[Y]/(Y^{d+1})$$

such that

$$v_*(a + bX) = daY^{d-1} + bY^d$$

Exercise 0.7. Try this for the general Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$, with $N = \binom{n+d}{d} - 1$, or at least try to show that

$$\big(v_{n,d}\big)_*(X^n) = Y^N, \quad \big(v_{n,d}\big)_*(1) = d^n Y^{d-1}$$

1. Correspondences

Definition 1.1. A *correspondence* from X to Y is any cycle of the cartesian product $X \times Y$. Since we will be interested mostly in correspondence classes, we adopt the notation

$$\operatorname{Corr}(X, Y) := C_\sim(X \times Y)$$

Example 1.2. Assume we are working modulo rational equivalence. By the Lemma of the previous section,

$$\operatorname{Corr}(\mathbb{P}^n, \mathbb{P}^m) \cong \mathbb{Q}[X, Y]/(X^{n+1}, Y^{m+1})$$

Correspondences are given a rather suggestive notation which leads us to believe that these are in some sense morphisms from X to Y; this is indeed the case. Our first definition toward this mode of thinking is the following:

Definition 1.3. Given $f \in \operatorname{Corr}(X_1, X_2)$, $g \in \operatorname{Corr}(X_2, X_3)$, we define the composition $g \circ f \in \operatorname{Corr}(X_1, X_3)$ as

$$g \circ f := \big(\pi_{13}\big)_* \big(\pi_{12}^*(f) \cdot \pi_{23}^*(g)\big)$$

where

$$\pi_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$$

is the standard projection.

Example 1.4. Consider 3 copies of $\mathbb{P}^1$, which we will denote $\mathbb{P}^1_X$, $\mathbb{P}^1_Y$, and $\mathbb{P}^1_Z$. Then,

$$\mathrm{Corr}(\mathbb{P}^1_X, \mathbb{P}^1_Y) \cong \mathbb{Q}[X, Y]/(X^2, Y^2), \quad \mathrm{Corr}(\mathbb{P}^1_Y, \mathbb{P}^1_Z) \cong \mathbb{Q}[Y, Z]/(Y^2, Z^2)$$

Under the above isomorphisms, we should be able to compute the composition $Y \circ X$. By definition,

$$Y \circ X = \big(\pi_{13}\big)_* \big(\pi_{12}^*(X) \cdot \pi_{23}^*(Y)\big)$$

Let us first focus on the term $\pi_{12}^*(X)$; as a class in $\mathbb{P}^1_X \times \mathbb{P}^1_Y$, X represents a line through the $\mathbb{P}^1_X$-axis, in the direction of the $\mathbb{P}^1_Y$ axis.

When we pull this back to $\mathbb{P}^1_X \times \mathbb{P}^1_Y \times \mathbb{P}^1_Z$, this acts like an inclusion mapping $X \mapsto X$. Note however that in $\mathbb{P}^1_X \times \mathbb{P}^1_Y \times \mathbb{P}^1_Z$, X has gained an extra degree of freedom in the $\mathbb{P}^1_Z$ direction, so it is more like a plane.

Similarly, for $\pi_{23}^*(Y)$, this pullback acts like an inclusion $Y \mapsto Y$. Again we have that Y has gained an extra degree of freedom in $\mathbb{P}^1_X$ direction, so it is a hyperplane in the product $\mathbb{P}^1_X \times \mathbb{P}^1_Y \times \mathbb{P}^1_Z$.

Using this, we are now looking at $\big(\pi_{13}\big)_* \big(XY\big)$, where the inside is simply the product of monomials. Note that XY represents the class of a codimension 2 surface with a full degree of freedom along the $\mathbb{P}^1_Z$ direction; when we project this onto $\mathbb{P}^1_X \times \mathbb{P}^1_Z$, we see that the class still retains a full degree of freedom in the $\mathbb{P}^1_Z$ direction, and has no degree of freedom in the $\mathbb{P}^1_X$ direction.

This is precisely the class of X in $\mathbb{P}^1_X \times \mathbb{P}^1_Z$, so we deduce that $Y \circ X = X$.

Example 1.5. Consider $1 \circ Y$ in the same setting as above. This simply becomes

$$(\pi_{13})_*(Y)$$

In $\mathbb{P}^1_X \times \mathbb{P}^1_Y \times \mathbb{P}^1_Z$, Y represented the class of a hyperplane with no degrees of freedom along $\mathbb{P}^1_Y$. When we project this on $\mathbb{P}^1_X \times \mathbb{P}^1_Z$, this covers the entire space, whose class is represented by 1. Thus $1 \circ Y = 1$. Note that this tells us that 1 is not necessarily the "identity correspondence", as one might expect.

Exercise 1.6. Consider

$$\mathrm{Corr}(\mathbb{P}^1_X, \mathbb{P}^1_Y) \cong \mathbb{Q}[X,Y]/(X^2,Y^2), \quad \mathrm{Corr}(\mathbb{P}^1_Y, \mathbb{P}^2_Z) \cong \mathbb{Q}[Y,Z]/(Y^2,Z^3)$$

Show that $Z^2 \circ Y = Z^2$ and $Z^2 \circ X = 0$.

2. Constructing a Category

The goal now is to construct a new category, the *category of correspondences*, where the morphisms are precisely the correspondences as defined above. We first need to see that correspondences satisfy the properties of morphisms; namely, associativity and the existence of an identity. We will employ the following, which is sometimes taken as an axiom or can be proved otherwise.

Proposition 2.1 (Projection formula)**.** *Given a morphism $\phi : X \to Y$, let $Z \in C(X)$, $Z' \in C(Y)$, we have*

$$\phi_*(Z \cdot \phi^*(Z')) = \phi_*(Z) \cdot Z'$$

Lemma 2.2. *Let $\Delta_X \in Corr(X,X)$ denote the class of the image of the diagonal map $\delta_X : X \to X \times X$. Then, for any $f \in Corr(X,Y)$*

and $g \in Corr(Y, X)$, we have

$$\Delta_X \circ g = g, \quad f \circ \Delta_X = f$$

Proof. We compute:

$$\begin{aligned} f \circ \Delta_X &= (\pi_{13})_*)(\pi_{12}^*(\Delta_X) \cdot \pi_{23}^*(f)) \\ &= (\pi_{13})_*((\Delta_X \times 1_Y) \cdot (1_X \times f)) \\ &= (\pi_{13})_*(\delta_X \times 1_Y)_*\big((1_X \times 1_Y) \cdot (\delta_X \times 1_Y)^*(1_X \times f)\big) \end{aligned}$$

Note that $1_X \times 1_Y$ acts as an identity element, so the above becomes:

$$\begin{aligned} (\pi_{13})_*(\delta_X \times 1_Y)_*\big((\delta_X \times 1_Y)^*(1_X \times f)\big) &= (\pi_{13})_*(1_X \times f) \\ &= f \end{aligned}$$

The case for $\Delta_X \circ g$ is essentially identical. □

Next, we have:

Lemma 2.3. *Composition of correspondences is associative.*

Proof. For convenience of notation, denote $f_{i,i+1}$ as the correspondence class of $\mathrm{Corr}(X_i, X_{i+1})$. The projections π_{ij} are denoted in the standard way. We see:

$$\begin{aligned} f_{34} \circ (f_{23} \circ f_{12}) &= (\pi_{14})_*\big(\pi_{13}^*(f_{23} \circ f_{12}) \cdot \pi_{34}^*(f_{34})\big) \\ &= (\pi_{14})_*\big(\pi_{13}^*((\pi_{13})_*(\pi_{23}^*(f_{23}) \cdot \pi_{12}^*(f_{12})) \cdot \pi_{34}^*(f_{34})\big) \\ &= (\pi_{14})_*\big(\big([(\pi_{13})_*(\pi_{23}^*(f_{23}) \cdot \pi_{12}^*(f_{12})] \times 1_{X_4}\big) \cdot (1_{X_1} \times f_{34})\big) \\ &= (\pi_{14})_*(\pi_{13} \times \mathrm{id})_*\big([(f_{12} \times 1_{X_3}) \cdot (1_{X_1} \times f_{23}) \times 1_{X_4}] \\ &\qquad \cdot (\pi_{13} \times \mathrm{id}_{X_4})^*(1_{X_1} \times f_{34})\big) \\ &= (\pi_{14})_*\big((f_{12} \times 1_{X_3} \times 1_{X_4}) \cdot (1_{X_1} \times f_{23} \times 1_{X_4}) \cdot (1_{X_1} \times 1_{X_2} \times f_{34})\big) \end{aligned}$$

Similarly, composing in the other order,

$$\begin{aligned} (f_{34} \circ f_{23}) \circ f_{12} &= (\pi_{14})_*\big(\pi_{12}^*(f_{12}) \cdot \pi_{24}^*(f_{34} \circ f_{23})\big) \\ &= (\pi_{14})_*\big((f_{12} \times 1_{X_3} \times 1_{X_4}) \cdot (1_{X_1} \times f_{23} \times 1_{X_4}) \cdot (1_{X_1} \times 1_{X_2} \times f_{34})\big) \end{aligned}$$

Thus the last line of both of the above computations are equal, in which case composition of correspondences is associative. □

Definition 2.4. Define the graph Γ_ϕ of a morphism $\phi : X \to Y$ as the composition

$$\Gamma_\phi = (\phi \times \mathrm{id}_Y) \circ \delta_Y : \ Y \xrightarrow{\delta_Y} Y \times Y \xrightarrow{\phi \times \mathrm{id}_Y} X \times Y$$

Define $h(\phi) := \Gamma_{\phi *}(1_Y)$; note that $h(\phi) \in \mathrm{Corr}(X, Y)$.

Observe that in the above definition, $h(\mathrm{id}_X) = \Delta_X$.

Definition 2.5. Define the category of correspondences $\mathrm{Corr}(k)$ to be the category whose objects consist of the objects of $\mathrm{Smproj}(k)$ and whose morphisms are precisely correspondences.

Theorem 2.6. *The association*

$$X \mapsto X, \quad \phi \mapsto h(\phi)$$

is a contravariant functor on Smproj(k) *to Corr*(k)

Proof. We only need show that for $\phi : Z \to Y$, $\psi : Y \to X$, that $h(\psi \circ \phi) = h(\phi) \circ h(\psi)$. Note that by definition,

$$\Gamma_{\psi \circ \phi} = \pi_{13}\big((\Gamma_\psi \times \mathrm{id}_Z) \circ \Gamma_\phi\big)$$

Whence

$$h(\psi \circ \phi) = \big(\pi_{13}\big) * \big((\Gamma_\psi \times \mathrm{id}_Z)_* \Gamma_{\phi *}(1_Z)\big)$$

Similarly, by definition of composition of correspondences,

$$h(\phi) \circ c(\psi) = (\pi_{13})_* \big((1_X \times \Gamma_{\phi *}(1_Z)) \cdot (\Gamma_{\psi *}(1_Y) \times 1_Z)\big)$$

In view of the above two expressions, our goal is to show that

$$(1_X \times \Gamma_{\phi *}(1_Z)) \cdot (\Gamma_{\psi *}(1_Y) \times 1_Z) = (\Gamma_\psi \times \mathrm{id}_Z)_* \Gamma_{\phi *}(1_Z)$$

We compute:

$$\begin{aligned}(1_X \times \Gamma_{\phi*}(1_Z)) \cdot (\Gamma_{\psi*}(1_Y) \times 1_Z) &= \big(1_X \times \Gamma_{\phi*}(1_Z)\big) \cdot \big(\Gamma_\psi \times \mathrm{id}_Z\big)_*(1_{Y\times Z}) \\ &= \big(\Gamma_\psi \times \mathrm{id}_Z\big)_*\Big(\big(\Gamma_\psi \times \mathrm{id}_Z\big)^*\big(1_X \times \Gamma_{\phi*}(1_Z)\big) \cdot 1_{Y\times Z}\Big) \\ &= \big(\Gamma_\psi \times \mathrm{id}_Z\big)_*\Big(\big(\Gamma_\psi \times \mathrm{id}_Z\big)^*\pi_{23}^*\big(\Gamma_{\phi*}(1_Z)\big)\Big)\end{aligned}$$

Then, note that $\pi_{23} \circ (\Gamma_\psi \times \mathrm{id}_Z) = \mathrm{id}_{Y\times Z}$. Thus, the above becomes

$$\big(\Gamma_\psi \times \mathrm{id}_Z\big)_*\Big(\big(\pi_{23} \circ (\Gamma_\psi \times \mathrm{id}_Z)\big)^*\big(\Gamma_{\phi*}(1_Z)\big)\Big) = \big(\Gamma_\psi \times \mathrm{id}_Z\big)_*\big(\Gamma_{\phi*}(1_Z)\big)$$

Which shows exactly what we want. Thus, $h(\psi \circ \phi) = h(\phi) \circ h(\psi)$ as contended. □

The importance of the above gives that we have "embedded" Smproj(k) into Corr(k), which is in fact an additive category over $\mathbb{Q}$. That is, all morphisms in Corr(k) are $\mathbb{Q}$-linear (or R-linear for any other ring) with respect to all arguments. The sum of varieties in Corr(k) is simply $X \oplus Y$, and we can define the tensor product as well:

Definition 2.7. Define the tensor product of objects in Corr(k) as $X \otimes Y := X \times Y$. We may also define the tensor product of morphisms for $f_1 \in \mathrm{Corr}(X_1, Y_1)$, $f_2 \in \mathrm{Corr}(X_2, Y_2)$:

$$f_1 \otimes f_2 := \tau_{23*}\big(\pi_{12}^*(f_1) \cdot \pi_{34}^*(f_2)\big)$$

where $\tau_{23} : X_1 \times Y_1 \times X_2 \times Y_2 \to X_1 \times X_2 \times Y_1 \times Y_2$ is the transposition switching the order of the middle two terms.

Proposition 2.8. *For $f_1,\ g_1 \in Corr(X_1, Y_1)$, $f_2,\ g_2 \in Corr(X_2, Y_2)$, we have:*

$$(f_1 \otimes f_2) \circ (f_1 \otimes g_2) = (f_1 \circ g_1) \otimes (f_2 \circ g_2)$$

Lemma 2.9 (Lieberman's Lemma)**.** *Let* $\alpha \in Corr(X, X')$, $\beta \in Corr(Y, Y')$, *and* $f \in Corr(X, Y)$. *Then,*

$$(\alpha \times \beta)_*(f) = \beta \circ f \circ^T \alpha$$