Geometric study of subfields of some non-commutative algebras

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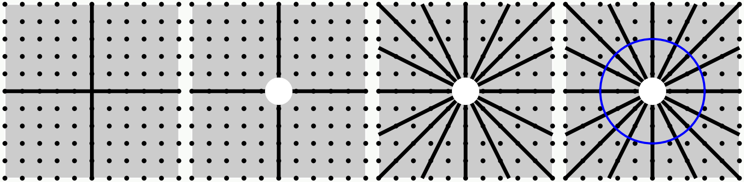
Matrices and projective space

Definition

Projective *n*-space is $\mathbb{P}^n_F = \left(\mathbb{A}^{n+1} \setminus \{0\}\right) / F^\times$. Points of projective space are thus given by $[a_0 : \cdots : a_n] \in \mathbb{A}^{n+1}$ such that

- not all of the a_i are 0
- $[a_0 : \cdots : a_n] = [\lambda a_0 : \cdots : \lambda a_n]$ for any $\lambda \in F^\times$.

Example: $\mathbb{P}^1_{\mathbb{R}} = S^1$



We can view $\mathbb{P}^1_F$ as $\mathbb{A}^1_F \cup \{\infty\}$:

$$[a : 1] \mapsto a \qquad [1 : 0] \mapsto \infty$$

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Proposition

There is a 1-1 correspondence between left ideals of dimension n in $M_n(F)$ and points in $\mathbb{P}_F^{n-1}$.

Example

- Let $J = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \mid a, b, c \in F \right\} \lhd_\ell M_3(F)$
- $\dim_F J = 3$
- Pick any element in J which has nonzero top row: $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$
- We associate to J the point $[1 : 0 : 0] \in \mathbb{P}_F^2$.

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Example

- Let $J = \left\{ \begin{pmatrix} a & 3a & 2a \\ b & 3b & 2b \\ c & 3c & 2c \end{pmatrix} \mid a, b, c \in F \right\} \lhd_\ell M_3(F)$
- Pick any element in J which has nonzero top row: $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- We associate to J the point $[1 : 3 : 2] \in \mathbb{P}^2_F$.
- Choosing a different element $\begin{pmatrix} 2 & 6 & 4 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix}$ we get $[2 : 6 : 4] = [2 \cdot 1 : 2 \cdot 3 : 2 \cdot 2] = [1 : 3 : 2]$.

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We may thus associate a geometric object to $M_n(F)$, and write

$$\mathbf{V}(M_n(F)) = \mathbb{P}_F^{n-1}.$$

Definition

A division algebra over F is an F-algebra such that every non-zero element is invertible. It is F-central if

$$Z(D) := \{a \in D \mid ab = ba \text{ for all } b \in D\} = F$$

A division algebra is thus a non-commutative version of a field, and are often called skew-fields.

Example (Hamilton's Quaternions)

$\mathbb{H} = \langle 1, i, j, k \mid i^2 = j^2 = -1, \text{ ij = -ji = k}\rangle$ is an $\mathbb{R}$-central division algebra of dimension 4.

Severi-Brauer varieties

A similar construction works for any algebra of the form $M_n(D)$.

Defintion

Let D be an F-division algebra. The Severi-Brauer variety associated to $M_n(D)$ is

$$\mathbf{V}(M_n(D)) = \left\{ J \lhd_\ell M_n(D) \mid \dim_F J = \sqrt{\dim_F M_n(D)} \right\}.$$

Example

- If $D = F$, we have $\mathbf{V}(M_n(D)) = \mathbb{P}_F^{n-1}$.
- If $D = \mathbb{H}$ and $n = 1$, we have

$$\mathbf{V}(D) = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}_{\mathbb{R}}^2.$$

Note this space has no points over the real numbers.

Severi-Brauer varieties

- $\mathbf{V}(M_n(D))$ is a twisted form of $\mathbb{P}^n$. If $\overline{F}$ is an algebraic closure of F, extending scalars yields

$$\mathbf{V}(M_n(D)) \otimes_F \overline{F} \cong \mathbb{P}^{n-1}_{\overline{F}}$$

- The points (of minimal degree n) of $\mathbf{V}(M_n(D))$ are in bijection with subfields $L \hookrightarrow M_n(D)$.

To study $M_n(D)$ and its arithmetic, one studies its subfields and how they fit together. The above construction gives a geometric object which parametrizes these subfields.

Geometry of points

Definition

Let X be an F-variety. The group of zero-cycles on X is the free abelian group generated by points of X. That is,

$$Z_0(X) = \left\{ \sum a_i p_i \mid a_i \in \mathbb{Z} \text{ and } p_i \in X \right\}.$$

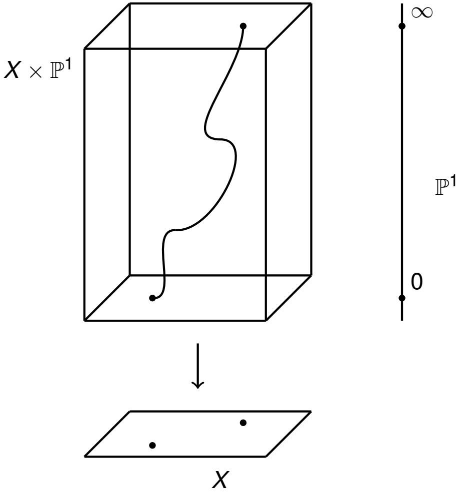
Two cycles α, β are equivalent if there exists a curve $C \subset X$, and a rational function $\frac{f}{g}$ on C such that

$$\alpha - \beta = \text{zeros}\left(\frac{f}{g}\right) - \text{poles}\left(\frac{f}{g}\right).$$

The Chow group of zero-cycles is then given by equivalence classes of zero-cycles

$$\mathrm{CH}_0(X) = Z_0(X)/\sim.$$

Geometry of points



Geometry of points

In a similar fashion, one considers the group of zero-cycles on X with coefficients that vary.

Construction

Let $X = \mathbf{V}(M_n(D))$.

- Points of X are in bijection with subfields of $M_n(D)$.
- For any $p \in X$, consider the corresponding subfield $L_p \hookrightarrow M_n(D)$.
- Let $\mathrm{CH}_0(X, K_1) = \{\sum(\lambda_p, p) \mid p \in X \text{ and } \lambda_p \in L_p\}$ and we identify elements which are equivalent (similar to the previous case).

The group $\mathrm{CH}_0(X, K_1)$ reflects arithmetic properties of $M_n(D)$ by using the arithmetic of its subfields.

Geometry of subfields

Theorem (Panin, '84)

Let D be a division algebra and $X = \mathbf{V}(M_n(D))$. There is a group isomorphism $\mathrm{CH}_0(X) \cong \mathbb{Z}$.

Through a geometric lens, all subfields $L \subseteq M_n(D)$ look the same.

Theorem (Merkurjev-Suslin, '92)

Let D be a division algebra and $X = \mathbf{V}(M_n(D))$. There is a group isomorphism $\mathrm{CH}_0(X, K_1) \cong D^\times/[D^\times, D^\times]$.

- The arithmetic of the subfields cannot be aligned as we move geometrically along the variety $\mathbf{V}(M_n(D))$.
- The misalignment can be measured by elements of $D^\times/[D^\times, D^\times]$.
- This reflects arithmetic complexity of both D and F.

Thank you!