THE BRAUER GROUP: A SURVEY

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Introduction

Notation/Conventions.

- F denotes a field.
- K denotes a finite field extension of F.
- F^s denotes the separable closure of F.
- If K/F is an extension, then we will usually denote $\mathrm{Gal}(K/F)$ by G.
- We will denote the absolute Galois group $\mathrm{Gal}(F^s/F)$ by Γ.

1. Brauer Group of a Field

Definition 1.1. Let F be a field. A *central simple algebra* A over F is a (finite-dimensional) F-algebra which is a simple ring such that $Z(A) = F$.

Examples.

- F-central division algebras.
- $M_n(F)$ (matrices over K).
- $\mathbb{H}$ (Hamilton's quaternions), an algebra over $\mathbb{R}$.
- Generalized quaternion algebras $(a, b)_F = \langle 1, i, j, ij \mid i^2 = j^2 = -1, ij = -ji\rangle$.

Theorem 1.2 (Wedderburn)**.** *Let A be a central simple algebra over F. Then $A \cong M_n(D)$ for some $n \geq 1$ and some F-central division algebra D.*

It turns out that knowing which algebras occur as central simple algebras over K provides a lot of information about the arithmetic structure of F. Thus, we consider the collection of all such F-algebras and investigate the properties of this collection. Let CSA_F denote the collection of isomorphism classes of central simple algebras over F.

Proposition 1.3. *If A, $B \in \mathrm{CSA}_F$ then $A \otimes_F B \in \mathrm{CSA}_F$.*

Proposition 1.4. *If $A \in \mathrm{CSA}_F$ then $A^{\mathrm{op}} := \{a^{\mathrm{op}} \mid a \in A, a^{\mathrm{op}} \cdot b^{\mathrm{op}} = ba\}$ is a central simple algebra over K. Furthermore, there is an isomorphism*

$$A \otimes_F A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_F(A)$$

given by $a \otimes b \mapsto (x \mapsto axb)$. Notice that the choice of a basis gives an isomorphism $\mathrm{End}_F(A)$ with a matrix algebra.

Proof. The above map is clearly a well-defined algebra homomorphism. Since $A \otimes A^{\mathrm{op}}$ is simple, this map is injective. A dimension count finishes the argument. $\square$

Remark. The proposition above states that $(\mathrm{CSA}_F, \otimes_F)$ forms an abelian semigroup. We will now endow this semigroup with an equivalence relation which in essence allows us to forget the difference between algebras which have the same underlying division algebra (which is the only nontrivial part of a central simple algebra).

Proposition 1.5. *Let $A, B \in \mathrm{CSA}_F$. The following are equivalent:*

(1) *$A \cong M_n(D)$, $B \cong M_m(C)$ and $D \cong C$.*
(2) *There is a division algebra D and integers m, n such that $A \cong M_n(D)$ and $B \cong M_m(D)$.*
(3) *There are positive integers m, n such that $M_n(A) \cong M_m(B)$.*

Definition 1.6. We say that $A, B \in \mathrm{CSA}_F$ are *Brauer equivalent* if they satisfy the equivalent conditions of the previous proposition. We write $A \sim_{\mathrm{Br}} B$.

Theorem 1.7. *The set $(\mathrm{CSA}_F \,/\sim_{Br})$ forms an abelian group under $\otimes_F$ called the **Brauer group** of F, denoted $\mathrm{Br}(F)$.*

Proof. We give a sketch.

- (Well-Defined) If $[A_1] = [A_2]$ and $[B_1] = [B_2]$ then A_i have underlying division algebra D and B_i have underlying division algebra D'. It follows that the product $[A_1] \cdot [B_1]$ and $[A_2] \cdot [B_2]$ will both be the class of $[D \otimes D']$
- (Identity) $[A] \cdot [F] = [A \otimes F] = [A]$.
- (Inverse) $[A] \cdot [A^{\mathrm{op}}] = [A \otimes A^{\mathrm{op}}] = [M_m(k)] = [k]$.

$\square$

Remark. Note that Brauer equivalence may be rephrased in terms of endomorphisms of F-modules: $A \sim_{\mathrm{Br}} B$ if and only if $M_n(A) \cong M_m(B)$. This is the case if and only if $A \otimes_F M_n(F) \cong B \otimes_F M_m(F)$. Since $M_n(F) = \mathrm{End}_F(F^n)$, we have that Brauer equivalence is equivalent to the condition that

$$A \otimes_F \mathrm{End}_F(P) \cong B \otimes_F \mathrm{End}_F(Q)$$

for $P = F^n$ and $Q = F^m$. This is exactly the notion of Brauer equivalence that will be used for commutative rings.

Aside. The data of a unitary algebra over F is given by a vector space A over F (say of dimension n), a distinguished element $1 \in A$ and a product $A \otimes_F A \to A$ such that restriction to $1 \otimes A$ and $A \otimes 1$ is the identity map. To give a product is to give a multiplication table. Fix a basis $\{u_i\}$. Then there are multiplication constants c_{ijk} given by $u_i u_j = \sum c_{ijk} u_k$. That A is associative is given by a set of relations in c_{ijk} which define a closed subscheme of $\mathrm{Spec}\, F[c_{ijk}]$. Call this subscheme Alg_n. The property of being a central simple algebra defines an irreducible subvariety of Alg_n. There is a norm function $N_A : A^* \to F^*$ on A. This is an analogue of the determinant and this norm is given by homogenous polynomial of degree n. To say that a given algebra (or point on our scheme) is a division algebra is equivalent to saying that N_A has a nontrivial zero, an arithmetic question. Thus, we see that central simplicity is a "geometric condition" (given by an irreducible subvariety) and being a division algebra is an "arithmetic" condition (proving existence of a nontrivial solution to some form; the study of quadratic forms for quaternion algebras). Since the Brauer forgets the full structure of a central simple algebra, only remembering the underlying division algebras, this group is keeping track of more arithmetic information about our field.

2. Brauer Group of a Commutative Ring

We only need an analogue of the notion of a central simple algebra and of Brauer equivalence.

Definition 2.1. Let R be a commutative ring and let A be an R-algebra. We say that A is *Azumaya* if it is a faithful, finitely generated projective R-module and

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an isomorphism.

Definition 2.2. Two Azumaya algebras A, B over R are *Brauer equivalent* if there are faithful, finitely generated projective R- modules P, Q such that

$$A \otimes_R \mathrm{End}_R(P) \cong B \otimes_R \mathrm{End}_R(Q).$$

We denote this by $A \sim_{\mathrm{Br}} B$. Let Az_R denote the collection of isomorphism classes of Azumaya algebras over R. We denote by $\mathrm{Br}(R) := (\mathrm{Az}_R / \sim_{\mathrm{Br}})$ the *Brauer group* of R.

3. Brauer Group of a Scheme

Let X be a scheme (over a field F, or over a scheme S). In a usual way, we extend a notion for commutative rings to schemes.

Definition 3.1. An *Azumaya algebra* $\mathcal{A}$ over X is a sheaf of $\mathcal{O}_X$-algebras such that for any open subset $U \subset X$ the $\mathcal{O}_X(U)$-algebra $\mathcal{A}(U)$ is Azumaya, i.e., satisfies the following conditions:

(1) $\mathcal{A}(U)$ is faithful
(2) $\mathcal{A}(U)$ is finitely generated
(3) $\mathcal{A}(U)$ is projective
(4) The $\mathcal{O}_X(U)$-algebra homomorphism $\mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{A}(U)^{\mathrm{op}} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{A}(U))$ is an isomorphism.

We extend our notion of Brauer equivalence to schemes by replacing faithful, finitely generated projective modules over a ring with locally free $\mathcal{O}_X$-modules of finite rank.

Definition 3.2. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ over X are *Brauer equivalent* if there exist locally free sheaves $\mathcal{L}, \mathcal{L}'$ of finite rank on X such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L}) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L}'),$$

written $\mathcal{A} \sim_{\mathrm{Br}} \mathcal{B}$. We let Az_X denote the collection of isomorphism classes of Azumaya X-algebras, and we define

$$\mathrm{Br}(X) := (\mathrm{Az}_X / \sim_{\mathrm{Br}})$$

4. Cohomology

4.1. **Nonabelian H^1.** Let G be a group and A a G-group (a group together with G-action). A 1-cocycle of G with values in A is a map $G \to A$, $\sigma \mapsto a_\sigma$ satisfying the *cocycle condition*

$$a_{\sigma\tau} = a_\sigma \sigma(a_\tau).$$

We say that two cocycles a_σ and b_σ are *cohomologous* if there is a $c \in A$ such that

$$a_\sigma = c^{-1} b_\sigma \sigma(c).$$

The first cohomology set $H^1(G, A)$ is the collection of 1-cocycles where we identify cohomologous cocycles. In general, this is not a group. It is a pointed set, with distinguished element $\sigma \mapsto 1$, the *trivial cocycle.*

4.2. **Twisted Forms.** Let A be a central simple algebra over F and let K/F be a finite Galois extension with Galois group G. We can associate to A a central simple algebra over K by extending scalars. We say that two central simple algebras A, B become isomorphic over K if there exists a K-isomorphism between A_K and B_K. We let $\mathrm{Tw}_K(A)$ denote the collection of (K/F)-twisted forms of A.

Given a F-automorphism $\sigma : K \to K$ (an element of G), we can tensor with A, obtaining a map $\sigma^A : A_K \to A_K$. Given any K-linear map $f : A_K \to B_K$ gives a new map $\sigma(f) : A_K \to B_K$ as follows:

$$A_K \xrightarrow{(\sigma^A)^{-1}} A_K \xrightarrow{f} B_K \xrightarrow{\sigma^B} B_K$$

This defines an action of G on $\mathrm{Aut}_K(A)$. If we are given two central simple algebras A and B and a K-isomorphism $g : A \to B$, we get a map $G \to \mathrm{Aut}_K(A)$, via $\sigma \mapsto a_\sigma := g^{-1} \circ \sigma(g)$. In fact, this is a 1-cocycle.

Theorem 4.1. *For any twisted form B of a central simple algebra B, the association $B \mapsto a_\sigma$ defines a bijection*

$$\mathrm{Tw}(A) \to H^1(G, \mathrm{Aut}_K(A))$$

By Wedderburn Theory, all central simple algebras are twisted forms of matrix algebras (so that all central simple algebras are twisted forms of each other). Thus, to obtain a nice description of the collection of all central simple algebras, we only need to determine the automorphism of group of $M_n(F)$.

Theorem 4.2 (Noether-Skolem)**.** *Over a field F, all automorphisms of $M_n(K)$ are inner, i.e., given by $M \mapsto CMC^{-1}$ for some invertible matrix C. In particular, $\mathrm{Aut}_F(M_n(F)) \cong PGL_n(F)$.*

Proof. We have a surjective map $GL_n(F) \to \mathrm{Aut}_F(M_n(F))$ via $C \mapsto \mathrm{Int}(C)$, conjugation by C. The kernel of this map is F^*, and the group $GL_n(F)/F^*$ is precisely $PGL_n(F)$. □

We let $\mathrm{CSA}_n(K)$ denote the collection of all central simple F-algebras of degree n which become matrix algebras over K.

Corollary 4.3. *There is a bijection* $\mathrm{CSA}_n(K) \to H^1(G, PGL_n(K))$*.*

Note that $H^1(G, PGL_n(K))$ classifies objects which are twisted forms of objects which have automorphism group $PGL_n(K)$. Are there other objects which have this same automorphism group? Yes, namely projective space $\mathbb{P}^{n-1}_K$. Thus, this cohomology set should classify varieties over F which are twisted forms of projective space (become isomorphic to projective space after base extension to K). Such varieties are called *Severi-Brauer varieties*.

Taking the limit over all n and all finite field extensions K/F, we obtain a new description of the Brauer group

$$\mathrm{Br}(F) \cong H^1(\Gamma, PGL_\infty(F^s)).$$

Aside. This definition tends to be a bit intractable, and we hope there is a simpler way of expressing the Brauer group cohomologically. Consider the short exact sequence of groups

$$1 \to K^* \to GL_n(K) \to PGL_n(K) \to 1.$$

We get an induced connecting homomorphism

$$\delta_n : H^1(G, PGL_n(K)) \to H^1(G, K^*).$$

Again, taking the union over all n and field extensions of F, we obtain an isomorphism

$$\mathrm{Br}(F) \cong H^2(\Gamma, (F^s)^*).$$

4.3. **Back to Schemes.** The above description of the Brauer group in terms of cohomology has a manifestation over schemes, but the story is not as clean. Going back to our description of the Brauer group in terms of $H^1(\Gamma, PGL_n(F^s))$ and Severi-Brauer varieties over F (twisted forms of projective space), we make an analogous definition using projective bundles.

We make the following substitutions:

$$\text{Galois Cohomology} \longleftrightarrow \text{Étale Cohomology}$$

$$\text{Severi-Brauer Varieties} \longleftrightarrow \text{Projective Bundles}$$

$$(F^s)^* \longleftrightarrow \mathbb{G}_m$$

For a scheme X, the cohomology set $H^1(X, PGL_n)$ is in bijective correspondence with the collection of projective bundles on X of rank $n-1$ (bundles whose geometric fibers are projective space of dimension $n-1$). Of course, there are those projective bundles which come from trivial bundles, namely projectivizations of vector bundles. We wish to consider such bundles as trivial. The cohomology set $H^1(X, GL_n)$ is in bijective correspondence with vector bundles. There is a map

$$H^1(X, GL_n) \to H^1(X, PGL_n)$$

taking a vector bundle to its projectivization. Taking the union over all quotients

$$H^1(X, PGL_n)/H^1(X, GL_n).$$

gives us $\mathrm{Br}(X)$ as defined above. In analogy with our above discussion concerning $H^2(\Gamma, (F^s)^*)$, we may also consider the cohomology set $H^2(X, \mathbb{G}_m)$, which is sometimes called the cohomological Brauer group. In general, there are inclusions

$$\mathrm{Br}(X) \subset H^2(X, \mathbb{G}_m)[\mathrm{tors}] \subset H^2(X, \mathbb{G}_m)$$

so that the Brauer group and cohomological Brauer group may not agree, as was the case over fields.

References

[1] P. L. Clark, On the Brauer Group-Notes for a Trivial Notions Seminar, unpublished, available at `http://math.uga.edu/~pete/trivial2003.pdf`.
[2] P. Gille, T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambridge University Press, 2006.
[3] D. Saltman, *Lectures on Division Algebras*, Regional Conference Series in Mathematics 94, 1999.