HOMOTOPICAL AND HOMOLOGICAL ALGEBRA

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1. Introduction

In the algebraic setting, one main method used in analyzing a given object is to investigate maps into and out of it. For example, the study of abelian categories consists of considering functors from a given abelian category into a well-behaved category such as **Ab**, the category of abelian groups and group homomorphisms. In this context, one property we can investigate is the *exactness* of such a functor.

One example of an abelian category is the category of modules over a ring Λ, denoted $\mathbf{Mod}(\Lambda)$. In fact, this is really the only example (see the Freyd-Mitchell Theorem or Mitchell's Embedding Theorem). In studying these categories, homological methods can be used to determine how far certain functors are from being exact. The objects which measure this non-exactness are called *derived functors*.

Unfortunately, these categories can be much too large to be the subject of a fruitful study (and may cause set-theoretic difficulties), so often our analysis must be reduced to the study of a more easily understood subcategory which still captures enough information to be useful. For example, in the algebraic K-theory of rings, to analyze a ring Λ, one restricts themselves to the study of finitely generated projective modules over Λ. This category is a much more tractable substitute for the full module category but still possesses a great deal of information about Λ.

Category $\mathcal{O}$ is a similar substitution for the module category $\mathbf{Mod}(U(\mathfrak{g}))$. We restrict our attention to those modules which have a certain finiteness condition so that many computations may be done explicitly. However, in restricting our attention to this much smaller category there is the possibility that we have lost many of the properties that allow us to study $\mathcal{O}$ by studying functors into and out of it. That is, we may by unable to use homological methods for our analysis.

Within the last half-century, homological algebra has been realized as a special case of homotopical algebra (to be made precise below). In this manuscript, we wish to establish some foundations of homotopical algebra and recognize how homological algebra occurs in particular situations. Our first goal is to make sense of the following:

Model Category $\mathscr{A}$	$\longleftrightarrow$	$C_b(\mathscr{A})$
Object of $\mathscr{A}$	$\longleftrightarrow$	Chain Complex
Weak Equivalence	$\longleftrightarrow$	Quasi-isomorphism
Fibrant Replacement	$\longleftrightarrow$	Injective Resolution
Cofibrant Replacement	$\longleftrightarrow$	Projective Resolution
Homotopy Category of $\mathscr{A}$	$\longleftrightarrow$	Derived Category of $\mathscr{A}$

Once we have introduced the homological methods (in the homotopical language), we then establish that these homological methods may be used to study category $\mathcal{O}$. We first check that $\mathcal{O}$ is abelian so that the concepts of kernel and image are at our disposal (in essence, this is all we need to define (co)homology). We then check that projective resolutions (cofibrant replacement) exist, and, using an interesting duality functor, that injective resolutions (fibrant replacement) also exist.

This exposition should be thought of as an outline of the theory and so proofs have not been included. However, all proofs regarding the theory of category $\mathcal{O}$ may be found in [2].

2. Homotopical Algebra

2.1. **Model Categories.** The theory of model categories allows us to precisely define the homotopy category of a given category so that we may utilize many concepts encountered in usual homotopy theory of topological spaces. However, for the purpose of studying derived functors on $\mathcal{O}$ or other abelian categories, we only wish to use homotopical algebra as a language and framework in which to develop homological methods in a natural way. We begin with some preliminary definitions.

Definition 2.1. A morphism $f : X \to Y$ in $\mathscr{C}$ is a *retract* of a morphism $g : Z \to T$ if there is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"'] \\ Y \arrow[r] & T \arrow[r] & Y \end{tikzcd}$$

where the horizontal compositions are identity morphisms.

Definition 2.2. Given a category $\mathscr{C}$, we define $\mathrm{Map}(\mathscr{C})$ to be the category whose objects are the morphisms of $\mathscr{C}$, and whose morphisms are commutative squares. A *functorial factorization* is an ordered pair (α, β) where $\alpha, \beta : \mathrm{Map}(\mathscr{C}) \to \mathrm{Map}(\mathscr{C})$ are functors such that for all objects f of $\mathrm{Map}(\mathscr{C})$, $f = \beta(f) \circ \alpha(f)$.

Definition 2.3. Given morphisms $i : A \to B$ and $p : X \to Y$ in a category $\mathscr{C}$, we say that i has the *left lifting property with respect to* p and p has the *right lifting property with respect* *to* i if for every commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

there is lift $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.

Definition 2.4. A *model structure* on a category $\mathscr{C}$ is the choice of three subcategories of morphisms of $\mathscr{C}$ (called weak equivalences, cofibrations and fibrations) and two functorial factorizations (α, β) and (γ, δ) which satisfy the following:

(1) If f and g are morphisms of $\mathscr{C}$ such that two of f, g and $g \circ f$ are weak equivalences, then so is the third.
(2) If f and g are morphisms of $\mathscr{C}$ such that f is a retract of g and g is a weak equivalence, cofibration or fibration, then so is f.
(3) A morphism is called a *trivial cofibration* (resp. *trivial fibration*) if it is both a cofibration (resp. fibration) and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to trivial fibrations.
(4) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

Definition 2.5. A *model category* is a category $\mathscr{C}$ with all small limits and colimits together with a model structure on $\mathscr{C}$.

Example 2.6. (Motivating example which is somewhat technical) The category **Top** of topological spaces and continuous maps has a model structure with weak equivalences given by weak equivalences (continuous maps which induce isomorphisms on homotopy groups), fibrations given by (Serre) fibrations (continous maps with the homotopy lifting property) and cofibrations given by cofibrations (maps which satisfy the homotopy extension property, the dual notion of the homotopy lifting property) which have the left lifting property with respect to acyclic Serre fibrations.

Example 2.7. (Example to be explained further below) Let $\mathscr{A}$ be an abelian category. The category $C_b(\mathscr{A})$ of bounded chain complexes in $\mathscr{A}$ can be given the structure of a model category, where weak equivalences are given by quasi-isomorphisms, cofibrations are given by chain maps which are monomorphisms in each degree and fibrations are given by chain maps which are epimorphisms in each degree and which have injective kernels. This is usually called the injective model structure on $C_b(\mathscr{A})$.

2.2. **The Homotopy Category.** For any model category $\mathscr{C}$, we can form the homotopy category $\mathrm{Ho}\,\mathscr{C}$ by "inverting" the weak equivalences. This is done in the category of topological spaces by defining

$$\mathrm{Hom}_{\mathrm{Ho}\,\mathbf{Top}}(X, Y) := \mathrm{Hom}_{\mathbf{Top}}(X, Y)/\simeq$$

where $\simeq$ denotes homotopy equivalence. In general, inverting weak equivalences is a bit more technical and we describe one such way below.

Let $\mathscr{C}$ be a model category and $\mathcal{W}$ the category of weak equivalences in $\mathscr{C}$. We form the free category $F(\mathscr{C}, \mathcal{W}^{-1})$ on the morphisms of $\mathscr{C}$ and the *reversals* of the arrows in $\mathcal{W}$.

Objects of $F(\mathscr{C}, \mathcal{W}^{-1})$ are the objects of $\mathscr{C}$ and a morphism is given by a finite collection of composable morphisms $(f_1, ..., f_n)$ where each f_i is either a morphism of $\mathscr{C}$ or the reversal w_i^{-1} of a morphism of $\mathcal{W}$. Composition is given by concatenation and the empty collection is the identity morphism. We then take the quotient of $F(\mathscr{C}, \mathcal{W}^{-1})$ by the following relations:

(1) $\mathrm{id}_A = (\mathrm{id}_A)$ for all objects A.
(2) $(f, g) = (f \circ g)$ for all composable pairs of morphisms.
(3) $\mathrm{id}_{\mathrm{dom}\, w} = (w, w^{-1})$ and $\mathrm{id}_{\mathrm{codom}\, w} = (w^{-1}, w)$ for all $w \in \mathcal{W}$.

The category $\mathrm{Ho}\,\mathscr{C}$ obtained from this process is called the *homotopy category of* $\mathscr{C}$.

Example 2.8. Consider the model category $C_b(\mathscr{A})$ for an abelian category $\mathscr{A}$. The homotopy category $\mathrm{Ho}\, C_b(\mathscr{A})$ is the category of bounded chain complexes where we invert quasi-isomorphism (that is, we identify complexes which have identical cohomology). The category obtained is denoted $D_b(\mathscr{A})$ and called the *bounded derived category of* $\mathscr{A}$. We will define this more precisely in the next section.

2.3. **(Co)Fibrant Replacement.** Given a model category $\mathscr{C}$, our assumption that $\mathscr{C}$ is *complete* (has all small limits) and *cocomplete* (has all small colimits) guarantees the existence of a terminal object 1 and an initial object $(*)$, obtained by taking the limit and colimit over the empty diagram. We call an object A of $\mathscr{C}$ *cofibrant* if the unique map $(*) \to A$ is a cofibration, and, dually, an object B is *fibrant* if the unique map $B \to 1$ is a fibration.

Let $f : (*) \to A$ be the unique map to A. Using our functorial factorization (α, β), we obtain a factorization of f as follows:

$$\begin{tikzcd} & QX \arrow[dr, "\beta(f)"] & \\ (*) \arrow[ur, "\alpha(f)"] \arrow[rr, "f"] & & X \end{tikzcd}$$

Here, QX is just an object of $\mathscr{C}$ which serves as the codomain of $\alpha(f)$ and domain of $\beta(f)$. Notice that by the axioms of model structure $\alpha(f)$ is a cofibration and $\beta(f)$ is a trivial fibration. Thus QX is a cofibrant object and QX and X are weakly equivalence so may be identified in the homotopy category. We called the assignment $X \mapsto QX$ the *cofibrant replacement functor*. Dually, we have a functor $X \mapsto RX$ called the *fibrant replacement functor*. This functor takes an object X of $\mathscr{C}$ and outputs a fibrant object RX which is weakly equivalent to X.

Example 2.9. Again, let us consider the model category $C_b(\mathscr{A})$. It can be shown that chain complexes consisting of injective objects of $\mathscr{A}$ are fibrant objects of $C_b(\mathscr{A})$. Similarly, chain complexes consisting of projective objects of $\mathscr{A}$ are cofibrant. Then the fibrant replacement functor sends a chain complex $C^\bullet$ to a fibrant complex which is quasi-isomorphic to $C^\bullet$.

Let A be an object of $\mathscr{A}$. We may regard A as object of $\mathscr{C}_b(\mathscr{A})$ which is A is degree 0 and the 0-object in every other degree. That is, there is a natural inclusion functor

$$\mathscr{A} \hookrightarrow C_b(\mathscr{A})$$

defined by

$$A \mapsto [\cdots \to 0 \to A \to 0 \to \cdots]$$

Then a fibrant replacement of A is usually called an *injective resolution of* A. Likewise, regarding $A \in \mathscr{A}$ as an object of $C_b(\mathscr{A})$, a cofibrant replacement is usually called a *projective resolution of* A.

3. Homological Algebra

We can realize homological algebra as a special case of the homotopical situation explored above. We will begin by describing our main objects of interest, derived functors, and fill in the definitions used the preceding section.

3.1. **Derived Categories.** Let $\mathscr{A}$ be an abelian category. A *chain complex* in $\mathscr{A}$ is a sequence of morphisms

$$\cdots \xrightarrow{\partial_{-n-1}} C_{-n} \xrightarrow{\partial_{-n}} \cdots \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} \cdots \xrightarrow{\partial_{n-1}} C_n \xrightarrow{\partial_n} \cdots$$

in $\mathscr{A}$ such that $\partial_{i+1} \circ \partial_i = 0$, i.e., $\operatorname{Im} \partial_i \subset \ker \partial_{i+1}$ for all i. We denote the above chain complex by $(C_\bullet, \partial_\bullet)$. A *chain map* $f : (C_\bullet, \partial_\bullet) \to (D_\bullet, \partial'_\bullet)$ is a sequence of morphisms $f_i : C_i \to D_i$ such that $f_{i+1} \circ \partial_i = \partial'_i \circ f_i$. Composition of chain maps is defined in the obvious way.

Let $C_b(\mathscr{A})$ be the category of bounded chain complexes and chain maps. That is, chain complexes $(C_\bullet, \partial_\bullet)$ such that $C_i = 0$ for $i \geq |n|$ for some $n \in \mathbb{Z}$ (Here, 0 denotes the zero object in $\mathscr{A}$). The *cohomology* of a chain complex is defined by $H^i(C_\bullet, \partial_\bullet) = \ker \partial_i / \operatorname{Im} \partial_{i-1}$. Taking cohomology of a complex defines a functor

$$H^i : C_b(\mathscr{A}) \to \mathbf{GrAb}$$

where $\mathbf{GrAb}$ is the category of graded abelian groups. Two chain complexes which have the same image under the cohomology functor are said to be *quasi-isomorphic.*

Recall from above that for an abelian category $\mathscr{A}$, the category $C_b(\mathscr{A})$ can be given the structure of a model category by taking weak equivalences to be quasi-isomorphisms, cofibrations to be degree-wise monomorphisms and fibrations to be degree-wise epimorphisms with injective kernels. The homotopy category $\operatorname{Ho} C_b(\mathscr{A})$ is called the *derived category of* $\mathscr{A}$, usually written $D_b(\mathscr{A})$.

3.2. **Derived Functors.** Here we present the basic set up for forming derived functors of a non-exact functor. We will treat the case of a covariant left-exact functor (such as $\operatorname{Hom}(M, -)$ for a fixed module M) using an *injective resolution.* The other cases (contravariant, right-exact and any combination of these) are defined similarly (and may necessitate the use of homology in place of cohomology as well as *projective* in place of injective).

Let $\mathscr{A}$ be an abelian category. An object I of $\mathscr{A}$ is said to be *injective* if for every morphism $f : A \to I$ and every morphism $g : A \to B$ there is a morphism $h : B \to I$ extending f. Diagramatically,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & I \arrow[dl, dotted, "\exists h"] \\ B & \end{tikzcd}$$

Equivalently, an object I is *injective* if $\operatorname{Hom}(-, I) : \mathscr{A} \to \mathbf{Ab}$ is an exact functor. We say $\mathscr{A}$ *has enough injectives* if for every object A of $\mathscr{A}$, there is an injection $A \hookrightarrow I$ where I is an injective object called the *injective hull* of A. The dual definition is that of a *projective* object and *projective cover* (simply reverse all of the above arrows).

If $\mathscr{A}$ is an abelian category with enough injectives, for any object A, we can form an *injective resolution* of A. This is an exact sequence

$$0 \to A \to I_0 \to I_1 \to \cdots \to I_n \to \cdots$$

of objects in $\mathscr{A}$ where I_i is injective and which we abbreviate by $A \to I_\bullet$.

Given a covariant left-exact functor $F : \mathscr{A} \to \mathbf{Ab}$, we form the right-derived functors R^iF as follows. Given an object A in $\mathscr{A}$, we fix an injective resolution $A \to I_\bullet$ and apply F:

$$0 \to F(I_0) \to F(I_1) \to \cdots \to F(I_n) \to \cdots$$

Since F is not right-exact, this gives a chain complex and the i^{th} cohomology group of this complex is the i^{th} derived functor of F applied to A, denoted $R^iF(A)$. That the groups $R^iF(A)$ are independent of the choice of injective resolution of A can be found in a multitude of sources (including [3]). Philosophically, homology and cohomology are homotopy-invariant so that any two fibrant or cofibrant replacements of an object are identified in the derived category (as they are both weakly equivalent to our original object in question).

Examples: Let Λ be a ring and let M, N be Λ-modules.

(1) $R^i \operatorname{Hom}_\Lambda(M, -) = \operatorname{Ext}^i_\Lambda(M, -)$
(2) $R^i \operatorname{Hom}_\Lambda(-, N) = \operatorname{Ext}^i_\Lambda(-, N)$
(3) $L_i(M \otimes_\Lambda -) = \operatorname{Tor}^\Lambda_i(M, -)$

4. Duality in Category $\mathcal{O}$

Now that we have established the general framework for utilizing homotopical/homological methods to study arbitrary (but nice enough) abelian categories, we will apply our new knowledge to category $\mathcal{O}$. As we have seen, category $\mathcal{O}$ is a more tractable category than $\mathbf{Mod}(U(\mathfrak{g}))$ (to be defined below) but one must verify that it retains many the desirable properties.

Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ the direct sum of positive root spaces [2]. Category $\mathcal{O}$ is defined to be the full subcategory of $\mathbf{Mod}(U(\mathfrak{g}))$ (the category of $U(\mathfrak{g})$-modules) consisting of modules M which satisfy the following conditions:

(1) M is finitely generated as a $U(\mathfrak{g})$-module.
(2) M is $\mathfrak{h}$-semisimple:
$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$$
(3) M is locally $\mathfrak{n}$-finite: for each $m \in M$, the subspace $U(\mathfrak{n}) \cdot m \subset M$ is finite dimensional.

By restricting our attention to this (much smaller) subcategory, we risk eliminating many useful properties of $\mathbf{Mod}(U(\mathfrak{g}))$, as categories of modules tend to be quite well-behaved. In particular, we hope that our new category is subject to analysis using methods of homological algebra. We begin with a result showing that this is indeed the case, to a certain extent.

Theorem 4.1. *Category $\mathcal{O}$ is an abelian category, closed under taking quotients, subobjects and finite direct sums.*

4.1. **Duality in $\mathcal{O}$.** Category $\mathcal{O}$ inherits a useful duality property which will allow us to use either fibrant or cofibrant replacement when computing higher derived functors, as we will see below. For a $U(\mathfrak{g})$-module M, there is a natural way of assigning a dual module M^*, arising from the Hopf algebra structure of $U(\mathfrak{g})$. For $x \in \mathfrak{g}$ and $f \in M^*$, we define the $U(\mathfrak{g})$-action to be

$$x \cdot f(v) = -f(x \cdot v).$$

However, it is not always the case that the dual module of a module in $\mathcal{O}$ also lies in $\mathcal{O}$, one of the drawbacks of restricting to a subcategory with a strong finiteness condition. We must thus modify our notion of dual so that it can be utilized in $\mathcal{O}$. This will come by "twisting" the action of $U(\mathfrak{g})$.

There is an anti-involution $\tau : \mathfrak{g} \to \mathfrak{g}$ defined as follows. For a semisimple Lie algebra $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, we write a basis for $\mathfrak{g}$ as $y_1, ..., y_m, h_1, ..., h_l, x_1, ..., x_m$, with $y_i \in \mathfrak{n}^-$, $h_i \in \mathfrak{h}$ and $x_i \in \mathfrak{n}$. Then τ is defined by interchanging x and y and fixing $\mathfrak{h}$. We define a new action of $\mathfrak{g}$ on M^* by

$$x \cdot f(v) = f(\tau(x) \cdot v).$$

This will be our default action on dual vector spaces.

Before defining the duality functor on $\mathcal{O}$, let us first consider the intermediate category $\mathcal{O} \subset \mathcal{C} \subset \mathbf{Mod}(U(\mathfrak{g}))$ of those $U(\mathfrak{g})$-modules which are weight modules with finite dimensional weight spaces. It can be shown [2, 3.3] that for a weight module M_λ, its dual $(M_\lambda)^*$ may be identified with the weight module $(M^*)_\lambda$ of the dual of M. Taking the dual of the weight modules "weight-wise," we can then define

$$M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^*,$$

called the *dual* of M in $\mathcal{C}$.

The duality functor enjoys many of the same useful properties as the usual duality functor on $U(\mathfrak{g})$ as we see in the next result. In particular, it restricts nicely to $\mathcal{O}$ and takes injective objects to projective objects.

Theorem 4.2. *The duality functor $M \mapsto M^\vee$ on $\mathcal{O}$ is exact and contravariant and satisfies the following properties:*

(1) *Duality induces a self-equivalence on $\mathcal{O}$: its square is naturally isomorphic to the identity functor.*
(2) *If $M, N \in \mathcal{O}$, then $(M \oplus N)^\vee \cong M^\vee \oplus N^\vee$, so that the duality functor preserves indecomposibility.*

By the exactness and contravariance of the duality functor, we see that if P is a projective object of $\mathcal{O}$, then $P^\vee$ is an injective object of $\mathcal{O}$. In our plight to use homological methods to study derived functors, it only remains to see that for any object M of $\mathcal{O}$, there is a projective or injective resolution of M so that we may use fibrant or cofibrant replacement in $C_b(\mathcal{O})$.

Theorem 4.3. *Category $\mathcal{O}$ has enough projectives (and therefore enough injectives, using duality).*

Thus, we may utilize our homotpical and homological machinery when studying category $\mathcal{O}$. Moreover, we may analyze the right or left derived functors of a functor of the form $F : \mathcal{O} \to \mathbf{Ab}$ regardless if it is covariant, contravariant, right-exact, left-exact or any combination of these.

Example 4.4. Since the duality functor gives us injective resolutions along with projective resolutions, we may compute $\mathrm{Ext}^n(M, N)$ in one of two ways. We may take a projective resolution of M (cofibrant replacement), apply the left-exact contravariant functor $\mathrm{Hom}_{\mathcal{O}}(-, N)$ and take the cohomology of the resulting chain complex. On the other hand, we may take an injective resolution of N (fibrant replacement), apply the functor $\mathrm{Hom}_{\mathcal{O}}(M, -)$ and take cohomology groups of the resulting chain complex.

References

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