Overarching goal: Embed the categ. $SmProj(k)$ into a category that is (close to being) abelian.

$SmProj(k)$ is too rigid and we first add more morphisms by way of generalized functions; these will be functions which don't pass the vertical line test. We also want to do this in a way that recovers usual morphisms.

To achieve this, we look to algebraic cycles.

<u>Algebraic</u> <u>cycles</u>:

Let k be a field, $SmProj(k)$ = cat. of smooth projective schemes $/k$.

A variety is a reduced scheme

Flashback to schemes:

A Weil divisor on X is a linear combination $D = \sum n_i Z_i$ where $n_i \in \mathbb{Z}$ and Z_i is a codim 1 subvariety of X. (irreducible)

$\mathrm{Div}(X)$ = group of divisors on X.

Def^{n} Let $f \in k(X)^*$ be a rational function on X. For each subvar. $V \subseteq X$, put $\mathrm{ord}(f) = \ell_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/(f))$, the length of the $\mathcal{O}_{X,V}$-module $\mathcal{O}_{X,V}/(f)$, defined as the longest chain of submodules in $\mathcal{O}_{X,V}/(f)$.

Ex: If X is nonsingular along V, $\mathcal{O}_{X,V}$ is a dvr, (since it is a regular local ring of dim 1).
and $\mathrm{ord}(f)$ is defined by the valuation.
(always satisfield for schemes regular in codim 1)

More precisely: $\nu_V : k(X)^* \longrightarrow \mathbb{Z}$, $\mathfrak{m}$ = reg. functions on X which vanish along V, and residue field $\mathcal{O}_{X,V}/\mathfrak{m} = k(V)$.

Ex: X curve$/k=\bar{k}$, $\mathrm{ord}_V(f) = \dim \mathcal{O}_{X,V}/(f)$.

Define $\mathrm{div}(f) \doteq \sum_{\substack{V \subseteq X \\ \mathrm{codim} V = 1}} \mathrm{ord}_V(f)[V]$.

One natural invariant of X is the divisor class group $Cl(X) = \mathrm{Div}(X)/\sim$ where $\sim$ denotes linear equivalence: divisors D, D' are equiv if $D - D' = \mathrm{div}(f)$ where $f \in K(X)^*$ (a rat'l function on X).

Ex: R Dedekind domain, $Cl(\mathrm{Spec} R) =$ ideal class group.

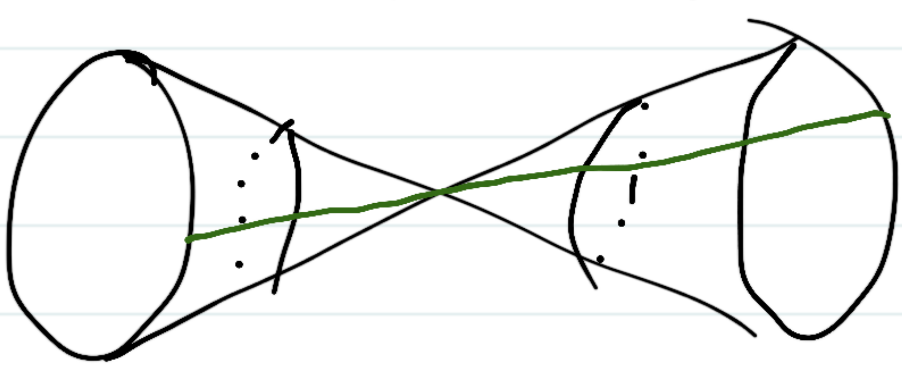
Ex: $Cl(\mathbb{A}^n) = 0$ (UFD's have trivial ideal class group)

Ex: $Cl(\mathbb{P}^n) = \mathbb{Z}$, generated by a hyperplane.

Ex: $H \subseteq \mathbb{P}^n$ hypersurface of degree d, then $Cl(\mathbb{P}^n - H) \cong \mathbb{Z}/d\mathbb{Z}$.

Ex: $X = \mathrm{Spec}\left(k[x,y,z]/(xy - z^2)\right)$

Then $Cl(X) = \mathbb{Z}/2$ generated by a ruling of the cone:



let's generalize this situation:

<u>Def^{n}</u>: A <u>prime i-cycle</u> on X is a irreducible subvariety of X of dim i. An <u>i-cycle</u> on X is a sum

$$\sum n_j [V_j].$$

where each V_j is an i-cycle on X.

- let $Z_i(X)$ be the collection of i-cycles on X. (i.e. the abelian group generated by the prime i-cycles).
- Similarly, we may define $Z^i(X)$ as the group generated by the codimension i cycles. Of course, $Z^i(X) = Z_{d-i}(X)$ where $d = \dim X$.

<u>Ex</u>: $Z^1(X) = Z_{d-1}(X) = \mathrm{Div}(X)$

<u>Ex</u>: $Z^d(X) = Z_0(X)$ admits a map to $\mathbb{Z}$ called the <u>degree map</u>, defined by:

$$\sum n_p [P] \longmapsto \sum n_p [k(P):k], \quad k(P) = \text{res. field at } P.$$

Coeffs: if we'd rather use ring of coeff other than $\mathbb{Z}$, define $Z_i(X)_F = Z_i(X) \otimes_{\mathbb{Z}} F$. Note $Z_0(X)_F \xrightarrow{\deg} F$.

Operations on cycles:

(Proper)

- Pushforward: let $f: X \longrightarrow Y$ be proper. If $V \subseteq X$ closed subvar., then $W = f(V)$ is closed subvar. of Y. This induces an inclusion $k(W) \subseteq k(V)$, a finite field extension when $\dim V = \dim W$. Let

$$\deg(V:W) = \begin{cases} [k(V):k(W)] & \text{if } \dim V = \dim W \\ 0 & \text{otherwise} \end{cases}$$

Define $f_*[V] = \deg(V:W)[W]$.

This gives a homomorphism $f_*: Z_i(X) \to Z_i(Y)$

- (Flat) Pullback: let $f: X \longrightarrow Y$ be <u>flat</u> of rel. dimension n.

<u>Ex</u>: - Open embedding $(n=0)$

- $X \times Y \longrightarrow X$ if Y is purely n-dim'l.
- Projection of $\mathbb{A}^n$ or projective bundle to the base.

Let $f: X \to Y$, $V \subseteq Y$ subvariety. Set $f^*[V] = [f^{-1}(V)]$
This defines a homomorphism

$$f^*: Z_i(Y) \longrightarrow Z_{i+n}(X)$$

- Intersection (Alex may give alternative formulation)

Let $V, W \subseteq X$. Let $\Delta: X \to X \times X$ be the diagonal morph.
Define $V \cdot W = \Delta^*(V \times W)$.

Note that this only defined for V, W meeting <u>properly</u>: if $\operatorname{codim} V = i$, $\operatorname{codim} W = j$, then $\operatorname{codim}(W \cap V) = i+j$ ($W \cap V$ is a union of Z_α and each Z_α has this codim).

This gives pairing $Z^i(X) \times Z^j(X) \longrightarrow Z^{i+j}(X)$.

Currently, our groups of cycles are superficially defined. We need to look for analogues of subspaces being homologous to one another. We look to endow the groups $Z_i(X)$ or $Z^i(X)$ with an equivalence relation which preserves the operations above (this will be the subject of Alex's talk).

For context, we say a few things about rational equivalence which mimics the description of the divisor class group.

Ex: (divisor of a rational function)

Let V be an $(i+1)$-dim'l subvar. of X, and let $f \in k(V)^*$ be a rational function on V. Define $[\mathrm{div}(f)] = \sum \mathrm{ord}_W(f)[W]$, where sum is taken over all $W \subseteq X$ of dim i.

This defines a homomorphism

$$\bigoplus_{\substack{V \subseteq X \\ \dim V = i+1}} k(V)^* \xrightarrow{\mathrm{div}} \bigoplus_{\substack{W \subseteq X \\ \dim W = i}} \mathbb{Z} = Z_i(X)$$

Just like with usual homology, we consider quotients (cycles modulo boundaries)

Def^n: An i-cycle $\alpha = \sum a_j V_j$ on X is <u>rationally equiv. to 0</u> if $\exists$ finitely many $(i+1)$-dim'l subvars $W_m \subseteq X$ and rat'l functions $f_m \in k(W_m)$ such that

$$\alpha = \sum [\mathrm{div}(f_m)].$$

Let $\mathrm{Rat}_i(X)$ be the group of rationally trivial i-cycles.

Define $CH_i(X) = Z_i(X) / \mathrm{Rat}_i(X)$

In other words,

$$CH_i(X) = Z_i(X)/\mathrm{im}(\mathrm{div})$$
$$= \mathrm{coker}\left(\mathrm{div}: \bigoplus k(V)^* \longrightarrow Z_i(X)\right)$$

Ex: $CH^1(X) = CH_{d-1}(X) = Cl(X) \underset{\uparrow}{=} \mathrm{Pic}(X)$

X smooth.

Ex: $p: E \to X$ affine bundle ($\exists \{U_\alpha\}$ cover of X so that $p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{A}^n$). Then $CH_i(X) \xrightarrow[p^*]{} \!\!\!\!\rightarrow CH_{i+n}(E)$

Ex:

$$CH_i(\mathbb{A}^n) = \begin{cases} 0 & \text{if } i<n \\ \mathbb{Z} & \text{if } i=n \end{cases}$$

Ex: $CH_i(\mathbb{P}^n) = \mathbb{Z}$ generated by L_i, linear subspace of dim i, $i = 0, \dots, n$.

The intersection product gives a (currently partially defined) pairing or product structure

$$CH^i(X) \times CH^j(X) \longrightarrow CH^{i+j}(X)$$

making $CH^*(X)$ into a ring. (we will need moving lemma or deformation to normal cone to fully define this structure).

Remark: (K-cohomology)

For any field F, we have algebraic K-groups $K_i(F)$ $i \geq 0$. In particular, $K_0(F) = \mathbb{Z}$, $K_1(F) = F^*$.

Thus, we can view div as a map on K-groups:

$$\mathrm{div}: \bigoplus_{\substack{V \subseteq X \\ \dim V = i+1}} K_1(k(V)) \longrightarrow \bigoplus_{\substack{W \subseteq X \\ \dim W = i}} K_0(k(W)) \longrightarrow 0$$

There are higher degree analogues of div, so this can be extended to a complex (residue hom's for DVR's)

$$\cdots \longrightarrow \bigoplus_{\substack{Y \subseteq X \\ \dim Y = i+1}} K_{p+i+1}(k(Y)) \longrightarrow \bigoplus_{\substack{V \subseteq X \\ \dim V = i}} K_{p+i}(k(V)) \longrightarrow \bigoplus_{\substack{W \subseteq X \\ \dim W = i-1}} K_{p+i-1}(k(W)) \rightarrow \cdots$$

Cohomology at middle term is $A_i(X, K_p)$ and we get Chow groups via

$$CH_i(X) = A_i(X, K_{-i})$$

This is the starting point of [EKM] in studying Chow groups, correspondences and Chow motives.
Gives better analogy for singular homology.