Adequate Equivalence Relations

Last time: Patrick only defined f^* for flat morphisms but used Δ^* to define intersection product (oops).

Given V, W subvarieties of X intersecting properly along $\bigcup Z_\alpha$

define $$i(V\cdot W; Z) := \sum_r (-1)^r \, \ell_{\mathcal{O}_{X,Z}}\left(\mathrm{Tor}_r^{\mathcal{O}_{X,Z}}(\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z})\right)$$

and the cycle $$V\cdot W := \sum_\alpha i(V\cdot W; Z_\alpha) Z_\alpha.$$

Remark $i(V\cdot W; Z) = 1$ when intersection is generically transverse.

A correspondence F from X to Y is a cycle in $X\times Y$.

Define $F(T) := (\pi_Y)_*(Z\cdot(T\times Y))$ for $T \in Z^*(X)$

when this makes sense. ${}^T F$ is just F in $Y\times X$.

For a morphism $f: X\to Y$, let $\Gamma_f \in X\times Y$ be the graph of f.

$f_*(Z) = F(Z)$

$f^*(Z) = {}^T F(Z)$ for $F = [\Gamma_f]$.

Now $V\cdot W = \Delta_X^*(V\times W)$ makes sense.

Definition

An <u>adequate equivalence relation</u> is a family of equivalence relations on $Z^*(X)$ such that:

1) compatible with grading and addition

2) $Z \sim 0$ on $X \Rightarrow Z \times Y \sim 0$ on $X \times Y$

3) $Z_1 \sim 0 \Rightarrow Z_1 \cdot Z_2 \sim 0$ (when defined)

4) $Z \sim 0$ on $X \times Y \Rightarrow (\pi_X)_*(Z) \sim 0$ on X

5) <u>Moving Lemma</u> Given $Z, W_1, \ldots, W_n \subseteq Z^*(X)$ have $Z' \sim Z$ such that $Z' \cdot W_1, \ldots, Z' \cdot W_n$ all defined.

Given $\sim$ define groups
$$Z^i_\sim(X) := \{Z \in Z^i(X) \mid Z \sim 0\}$$
$$C^i_\sim(X) = Z^i(X)/Z^i_\sim(X).$$

<u>Lemma</u> (1) $C^*_\sim(X)$ is a commutative ring under intersection product.

(2) $f: X \to Y$ in $\mathrm{SmProj}(k)$ induces

$f_*: C^*_\sim(X) \to C^*_\sim(Y)$ a group hom. (grading changes!)

$f^*: C^*_\sim(Y) \to C^*_\sim(X)$ a graded ring hom.

Suppose $Z \in Z^i(X)$.

Rational Equivalence

$Z \sim_{rat} 0$: there exists $W \in Z^i(\mathbb{P}^1 \times X)$ such that $W(0) = Z$, $W(\infty) = 0$.

Algebraic Equivalence

$Z \sim_{alg} 0$: there is a curve C with points a, b and there exists $W \in Z^i(C \times X)$ such that $W(a) = Z$, $W(b) = 0$.

Smash Nilpotent Equivalence

$Z \sim_{\otimes} 0$: $\underbrace{Z \times \cdots \times Z}_{n \text{ times}} \sim_{rat} 0$ on $\underbrace{X \times \cdots \times X}_{n \text{ times}}$ for some n

Numerical Equivalence

$Z \sim_{num} 0$: $\deg(Z \cdot W) = 0$ for all $W \in Z^{\dim X - i}(X)$ such that this makes sense

Homological Equivalence

Let H^* be a Weil cohomology theory with cycle class map $\gamma_X : Z^d(X) \to H^{2d}(X)$.
(e.g. singular cohomology of X_{an} with $\mathbb{Q}$-coefs, de Rham cohomology, étale cohomology, crystalline cohomology)

$Z \sim_{hom} 0$: $\gamma_X(Z) = 0$

Standard Conjecture D says this is independent of chosen cohomology theory.

Known

$$\begin{tikzcd} \mathrm{rat} \arrow[r, "\neq"] \arrow[dr] & \mathrm{alg} \arrow[r, "\neq"] \arrow[d, dotted, "\mathbb{Q}\text{-coefs}"] & \mathrm{hom} \arrow[r] & \mathrm{num} \\ & \otimes \arrow[ur] & & \end{tikzcd}$$

(many flavors)

(Finest) (Coarsest)

Conjectural $\quad \mathrm{rat} \to \mathrm{alg} \to \boxed{\otimes = \mathrm{hom} = \mathrm{num}}$ if $k = \bar{k}$

Def The **Chow groups** are $CH^i(X) := Z^i(X)/Z^i_{rat}(X)$.

The **Chow ring** is $CH^*(X)$.

Remarks $(k = \bar{k})$

- $CH^1(X) = Z^1(X)/Z^1_{rat}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X) = \mathrm{Pic}(X)$
- $Z^1_{alg}(X)/Z^1_{rat}(X) \cong \mathrm{Pic}^0_{red}(X)(k)$
- $NS(X) := Z^1(X)/Z^1_{alg}(X)$ is fin. gen. abelian group (Néron-Severi group)
- $Z^1_{hom}(X) = Z^1_{num}(X) = \{ D \in Z^1(X) : nD \sim_{alg} 0 \text{ for some } n \}$
- $C^i_{alg}(X)$ is countable.
- $\mathrm{Griff}^i(X) := Z^i_{hom}(X)/Z^i_{alg}(X)$ (can be ∞-dim'l $\mathbb{Q}$-vs when $\dim X = 3$, $i = 2$)
- $Nm^i(X) := C^i_{num}(X)_{\mathbb{Q}}$ is $\mathbb{Q}$-vector space of dim $\leq \dim_{\mathbb{Q}_\ell} H^{2i}_{\acute{e}t}(X)$