

Adequate Equivalence Relations

Last time: Patrick only defined f^* for flat morphisms
but used Δ^* to define intersection product (cobs).

Given V, W subvarieties of X intersecting properly along VZ_α

$$\text{define } i(V \cdot W; z) := \sum_r (-1)^r l_{\theta_{X,z}}^r (\text{Torr}_r^{\theta_{X,Z}}(\theta_{V,z}, \theta_{W,z}))$$

$$\text{and the cycle } V \cdot W := \sum_\alpha i(V \cdot W; Z_\alpha) | Z_\alpha.$$

Remark $i(V \cdot W; z) = 1$ when intersection is generically transverse.

A correspondence F from X to Y is a cycle in $X \times Y$.

$$\text{Define } F(T) := (\pi_Y)_*(Z \cdot (T \times Y)) \text{ for } T \in Z^*(X)$$

when this makes sense. T_F is just F in $Y \times X$.

For a morphism $f: X \rightarrow Y$, let $T_f \in X \times Y$ be the graph of f .

$$f_*(z) = F(z) \quad \text{for } F = [T_f].$$

$$F^*(z) = T_F(z)$$

Now $V \cdot W = \Delta_X^*(V \times W)$ makes sense.

Definition

A n adequate equivalence relation is a family of equivalence relations on $Z^*(k)$ such that:

1) compatible with grading and addition

2) $Z \sim 0$ on $X \Rightarrow Z \times Y \sim 0$ on $X \times Y$

3) $Z_1 \sim 0 \Rightarrow Z_1 \cdot Z_2 \sim 0$ (when defined)

4) $Z \sim 0$ on $X \times Y \Rightarrow (\gamma_X)_*(Z) \sim 0$ on X

5) Moving Lemma Given $Z, W_1, \dots, W_n \subseteq Z^*(k)$

have $Z' \sim Z$ such that $Z', W_1, \dots, Z' \cdot W_n$ all defined.

Given n define groups $Z_n^i(k) := \{Z \in Z^i(k) \mid Z \sim 0\}$

$$C_n^i(k) = Z^i(k)/Z_n^i(k).$$

Lemma

(1) $C_n^*(k)$ is a commutative ring under intersection product.

(2) $f: X \rightarrow Y$ in $\text{SmProj}(k)$ induces

$f_*: C_n^*(X) \rightarrow C_n^*(Y)$ a group hom. (grading changes!)

$F^*: C_n^*(Y) \rightarrow C_n^*(X)$ a graded ring hom.

Suppose $Z \in Z^i(X)$.

Rational Equivalence

$Z \sim_{rat} 0$: there exists $W \in Z^i(\mathbb{P}^1 \times X)$

such that $w(0) = Z, w(\infty) = 0$.

Algebraic Equivalence

$Z \sim_{alg} 0$: there is a curve C with points a, b

and there exists $W \in Z^i(C \times X)$

such that $w(a) = Z, w(b) = 0$.

Smash Nilpotent Equivalence

$Z \sim_{\otimes} 0$: $\underbrace{Z \times \dots \times Z}_{n \text{ times}} \sim_{rat} 0$ on $\underbrace{X \times \dots \times X}_{n \text{ times}}$

for some n

Numerical Equivalence

$Z \sim_{num} 0$: $\deg(Z \cdot W) = 0$ for all $W \in Z^{\dim X - i}(X)$
such that this makes sense

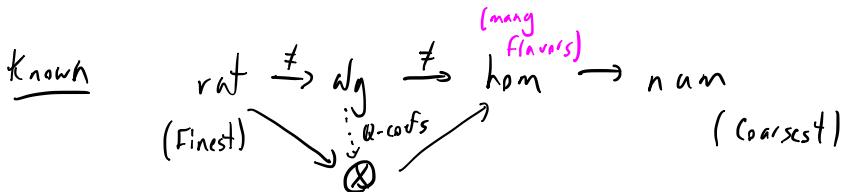
Homological Equivalence

Let H^* be a Weil cohomology theory with cycle class map $\gamma_X: Z^d(X) \rightarrow H^{2d}(X)$.

(e.g. singular cohomology of X with \mathbb{Q} -coeffs, de Rham cohomology,
étale cohomology, crystalline cohomology)

$Z \sim_{hom} 0$: $\gamma_X(Z) = 0$

Standard Conjecture D
says this is independent of
chosen cohomology theory.



Conjectural $\text{rat} \rightarrow \text{alg} \rightarrow \boxed{\otimes = \text{hom} = \text{num}}$ if $k = \bar{k}$

Def The Chow groups are $CH^i(X) := Z^i(X)/Z_{\text{rat}}^i(X)$.
 The Chow ring is $CH^*(X)$.

Remarks ($k = \bar{k}$)

- $CH^i(X) = Z^i(X)/Z_{\text{rat}}^i(X) = \frac{\text{Div}(X)}{\text{PDiv}(X)} = \text{Pic}(X)$
- $Z_{\text{alg}}^i(X)/Z_{\text{rat}}^i(X) \cong \text{Pic}_{\text{red}}^0(X)(k)$
- $NS(X) := Z^i(X)/Z_{\text{alg}}^i(X)$ is fin. gen. abelian group
(Néron-Severi group)
- $Z_{\text{hom}}^i(X) = Z_{\text{num}}^i(X) = \{O \in Z^i(X) : nO \sim_{\text{alg}} O \text{ torsion}\}$
- $C_{\text{alg}}^i(X)$ is countable.
- $\text{Griff}^i(X) := Z_{\text{hom}}^i(X)/Z_{\text{alg}}^i(X)$ (can be ∞ -dim'l \mathbb{Q} -vs)
(when $\dim X = 3, i = 2$)
- $N_m^i(X) := C_{\text{num}}^i(X)_{\mathbb{Q}}$ is \mathbb{Q} -vector space of $\dim \leq \dim_{\mathbb{Q}_p} H^{2i}_{\text{ét}}(X)$