Notes from the UGA $\mathbb{A}^1$-Homotopy Seminar

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Contents

1 Categories, Functors, Natural Transformations and Presheaves

1.1 Categories

Definition 1.1. A *category* $\mathscr{C}$ consists of the following data:
(C1) a collection of objects, $\mathrm{Ob}(\mathscr{C})$
(C2) for every pair A, $B \in \mathrm{Ob}(\mathscr{C})$ a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* from A to B
(C3) for each $A \in \mathrm{Ob}(\mathscr{C})$ a distinguished morphism $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$
(C4) for every three objects A, B, $C \in \mathrm{Ob}(\mathscr{C})$ a function "composition"

$$\mathrm{Hom}_{\mathscr{C}}(B, C) \times \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$$

$$(f, g) \mapsto f \circ g = fg$$

satisfying the following axioms:
(A1) *Associativity*: If

$$A \xrightarrow{h} B \xrightarrow{g} C \xrightarrow{f} D\ ,$$

then $(fg)h = f(gh)$.
(A2) *Identity*: If

$$A \xrightarrow{f} B\ ,$$

then $f1_A = f$ and $1_B f = f$.

The utility of the above definition is that there are plenty of examples to consider. Some of the most common examples are given below.

1.1.1 Examples

• Let $\mathscr{C}$ be a category with $\mathrm{Ob}(\mathscr{C}) = \{*\}$ and $\mathrm{Hom}_{\mathscr{C}}(*, *) = 1_*$.

• Let $\mathscr{C} = \mathbf{Sets}$. Then $\mathrm{Ob}(\mathscr{C}) =$ "all sets," (or at least all sets in some fixed universe) and $\mathrm{Hom}_{\mathscr{C}}(A, B) = \{f : A \to B\}$, the set of functions from A to B.

In a sense, categories arise in two different flavors, big or small. The above example is that of a big category. The following example is a small category.

• Fix a set S. Define the category $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{C}) = \{T \mid T \subseteq S\}$ and $\mathrm{Hom}_{\mathscr{C}}(A, B) = \{f : A \hookrightarrow B\}$.

• Let G be a group. Let $\mathscr{C}_G$ be the category with $\mathrm{Ob}(\mathscr{C}_G) = \{*\}$ and $\mathrm{Hom}(*, *) = G$, the underlying set of G. Also, $1_* = \mathrm{id}_G$ and the composition function is simply the group law of G.

The above example works equally well in the case that G is only a monoid.

• Let $\mathscr{C} = \mathbf{Top}$. Then $\mathrm{Ob}(\mathscr{C})$ = all topological spaces and $\mathrm{Hom}_{\mathscr{C}}(X, Y) = \{f : X \to Y \mid f$ is continuous$\}$.

• Let X be a topological space. Let $\mathbf{Op}(X)$ be the category of open sets of X. Then $\mathrm{Ob}(\mathbf{Op}(X)) = \{U \subseteq X \mid U \text{ is open}\}$ and $\mathrm{Hom}_{\mathbf{Op}(X)}(A, B)$ = inclusions.

Definition 1.2. Let $\mathscr{C}$ be a category and let A, $B \in \mathrm{Ob}(\mathscr{C})$. We say A *is isomorphic to* B, written $A \cong_{\mathscr{C}} B$ if there exists a morphism $f : A \to B$ and a morphism $g : B \to A$ such that $fg = 1_B$ and $gf = 1_A$.

Categories are a particular example of an object which has a certain structure. Naturally, one can define maps between categories which preserve this categorical structure. These maps are called functors.

1.2 Functors

Definition 1.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ is a rule $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ together with, for each pair $A, B \in \mathrm{Ob}(\mathscr{C})$ a function $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ satisfying the following axioms:
(F1) $F(fg) = F(f)F(g)$.
(F2) $F(1_A) = 1_{F(A)}$.

Definition 1.4. A *contravariant functor* is as above with the alteration

$$F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(B), F(A)).$$

Remark 1.5. If $\mathscr{C}$ is a category we can define $\mathscr{C}^{\mathrm{op}}$ as $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$ and $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. Thus, we may think of a contravariant functor $F : \mathscr{C} \to \mathscr{D}$ as a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Functors will turn out to be our main object of study. We will begin with a category in which we are interested, such as **Schemes** or **Top**, and we will replace it by a new category whose objects are functors taking objects of our original category to the category of sets. To gain a familiarity with functors, we begin with a few examples.

1.2.1 Examples

• F : **Groups** $\to$ **Sets**. F takes a group and maps it to its underlying set. It forgets its group structure and we call it a *forgetful functor*.

• F : **Sets** $\to$ **Groups** via $S \mapsto \langle S \rangle$, the free group generated by S.

• $\pi_1 : \mathbf{Top}_* \to \mathbf{Groups}$. Here, $\mathrm{Ob}(\mathbf{Top}_*)$ = all pointed spaces.

• **HoTop**. $\mathrm{Ob}(\mathbf{HoTop}) = \mathrm{Ob}(\mathbf{Top})$ and $\mathrm{Hom}_{\mathbf{HoTop}}(X, Y) = \mathrm{Hom}_{\mathbf{Top}}(X, Y)/\sim$, where $\sim$ denotes homotopy equivalence.

• $H_*(-) : \mathbf{Top} \to \mathbf{Ab}$ or $H^*(-) : \mathbf{Top} \to \mathbf{Ab}$, the homology and cohomology groups of a topological space. We may also realize $H^*(X)$ as $\mathrm{Hom}_{\mathbf{HoTop}}(X, K(\mathbb{Z}, *))$, where $K(\mathbb{Z}, *)$ is the Eilenberg-MacLane space.

• Let $\mathscr{C}$ be any category and $A \in \mathrm{Ob}(\mathscr{C})$. We can define a contravariant functor $F : \mathscr{C} \to \mathbf{Sets}$ by $B \mapsto \mathrm{Hom}_{\mathscr{C}}(B, A)$. That is, associated to every object in $\mathscr{C}$ is a contravariant functor. It has the property that given a map $B \to C$, there is a natural way of taking a morphism $C \to A$ to a morphism $B \to A$ via composition:

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & A \\ & C \arrow[ur] & \end{tikzcd}$$

Thus, for each object A, we may associate the functor $F_A : \mathscr{C} \to \mathbf{Sets}$ via $F_A(B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. If $f : B \to C$ then we have a map $F_A(f) : F_A(C) \to F_A(B)$ given by $F_A(f)(g) = gf$ for $g \in \mathrm{Hom}_{\mathscr{C}}(C, A)$.

The above example is the most important example given, as we will encounter it again and again. Using this construction, we may take objects in a category of interest and reinterpret them as functors or to obtain functors associated to our objects.

Definition 1.6. Functors of the form

$$F_A : \mathscr{C} \to \mathbf{Sets}$$

are called *representable* with *representing object* A.

Definition 1.7. A *presheaf* is any contravariant functor $\mathscr{C} \to \mathbf{Sets}$

1.3 Natural Transformations

Definition 1.8. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation* $\alpha : F \to G$ is a rule which associates to each $A \in \mathscr{C}$ a morphism $\alpha(A) : F(A) \to G(A)$ such that for all $f : A \to B$

$$G(f)\alpha(A) = \alpha(B)F(f).$$

That is, the following diagram is commutative:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha(B)"] & G(B) \end{tikzcd}$$

2 Presheaves and Limits

Recall that if $\mathscr{C}$ and $\mathscr{D}$ are categories and $F, G : \mathscr{C} \to \mathscr{D}$ are functors, a natural transformation $T : F \to G$ is a collection of maps $T_A : F(A) \to G(A)$ for each $A \in \mathrm{Ob}(\mathscr{C})$ such that the following diagram commutes:

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & G(B) \arrow[d] \\ F(A) \arrow[r] & G(A) \end{tikzcd}$$

We observe that $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ is a category with $\mathrm{Ob}(\mathbf{Fun}(\mathscr{C}, \mathscr{D})) = \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ and $\mathrm{Hom}_{\mathbf{Fun}(\mathscr{C},\mathscr{D})}(F, G) = \mathrm{Nat}(F, G)$, natural transformations from F to G.

2.1 Equivalence of Categories

Definition 2.1. Let $F : \mathscr{C} \to \mathscr{D}$. We say that F is *faithful* if for all $A, B \in \mathrm{Ob}(\mathscr{C})$,

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

is injective. F is *full* if

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

is surjective. If F is both full and faithful, we say that F is *fully faithful.* The functor F is *essentially surjective* if for all $A \in \mathrm{Ob}(\mathscr{D})$, there exists $B \in \mathrm{Ob}(\mathscr{C})$ such that $F(B) \cong_{\mathscr{D}} A$.

Definition 2.2. A functor F is an *equivalence* if it is fully faithful and essentially surjective.

2.1.1 Examples

• Let $\mathscr{C}$ be the category of all finite subsets of $\mathbb{N}$. Let $\mathscr{D}$ be the category of finite sets. Then $F : \mathscr{C} \to \mathscr{D}$ given by inclusion is an equivalence of categories.

• $\mathrm{Spec} : \mathbf{CommRings}^{\mathrm{op}} \to \mathbf{AffSch}$ defined by $R \mapsto \mathrm{Spec}\, R$ is an equivalence of categories.

We now give an alternative definition for an equivalence of categories.

Definition 2.3. A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G \cong_{\mathbf{Fun}(\mathscr{D},\mathscr{D})} 1_{\mathscr{D}}$ and $G \circ F \cong_{\mathbf{Fun}(\mathscr{C},\mathscr{C})} 1_{\mathscr{C}}$.

Recall that if $\mathscr{C}$ is a category, the presheaves on $\mathscr{C}$, $\mathcal{P}re(\mathscr{C}) = \mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$, contravariant functors from $\mathscr{C}$ to **Sets**. Given $A \in \mathrm{Ob}(\mathscr{C})$ we obtain $F_A \in \mathrm{Ob}(\mathcal{P}re(\mathscr{C}))$, where $F_A(B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. This gives a functor $F : \mathscr{C} \to \mathcal{P}re(\mathscr{C})$. We will see that this functor is in fact fully faithful, a consequence of Yoneda's Lemma, but is not necessarily essentially surjective.

Example 2.4. Let $\mathscr{C} = \mathbf{Top}$. Given a topological space X, we can, in a sense, reconstruct it by considering maps $\mathrm{Hom}_{\mathscr{C}}(*, X) =$ "X" (the underlying set of X). Using the functor F_X, we can recover X by looking at the value of F_X on a single point, $F_X(*)$. One can think of $\mathrm{Hom}_{\mathscr{C}}(*, X) = F_X(*) =$ as the "points of X," and should think of $F_X(Z)$ as continuous families of points in X parameterized by $Z \subset X$, or perhaps a Z-valued point of X.

The idea of Yoneda's Lemma is the observation that if we have topological spaces X and Y and a continuous map $f : X \to Y$, this gives a way of taking Z-valued points of X to Z-valued points of Y, $F_X(Z) \to F_Y(Z)$. Occasionally, the notation $X(Z)$ is used to denote $F_X(Z)$. If we have maps of Z valued points for every Z, we can use this information to recover f. Suppose $T : F_X \to F_Y$ is a natural transformation. Consider $T(X) : F_X(X) \to F_Y(X)$. But 1_X is an X-valued point of X and $F_Y(X) = \mathrm{Hom}_{\mathbf{Top}}(X, Y)$. The image of 1_X will give a map from X to Y. It turns out that the two above constructions are inverses of each other. That is, if we start out with a continuous map we have a corresponding map between the functors F_X and F_Y. Looking at what this natural transformation does on X-valued points, we recover our original map. Likewise, if we begin with a natural transformation between two functors and consider the image of the identity map on X, we obtain a continuous map such that the corresponding map of functors is the natural transformation we started with.

Lemma 2.5. *If $A \in \mathrm{Ob}(\mathscr{C})$ and G is a presheaf on $\mathscr{C}$, then $G(A) = \mathrm{Hom}_{\mathcal{P}re(\mathscr{C})}(F_A, G)$.*

Proof. If $g \in G(A)$, we want a map $F_A(B) \to G(B)$. Let $f \in F_A(B)$. Then

$$f : B \to A.$$

Applying G, we have

$$G(f) : G(A) \to G(B).$$

Since $g \in G(A)$, $f \mapsto G(f)(g)$. Thus, $f \mapsto G(f)(g)$ defines a map corresponding to the element $g \in G(A)$. Conversely, given $T : F_A \to G$. $1_A \in F_A(A)$. $T_A(1_A) \in G(A)$. Thus, $T_A(1_A)$ is the element of $G(A)$ corresponding to T. $\square$

Example 2.6. Let $G = F_B$. Then $F_B(A) = \mathrm{Hom}_{\mathcal{P}re(\mathscr{C})}(F_A, F_B)$. But $F_B(A) = \mathrm{Hom}_{\mathscr{C}}(A, B)$. Thus, we obtain the same identification as in the discussion above: maps from A to B are the same as maps on the corresponding presheaves.

In the above example, since we have $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathcal{P}re(\mathscr{C})}(F_A, F_B)$ we can see that the natural map $\mathscr{C} \to \mathcal{P}re(\mathscr{C})$ is fully faithful.

The general way that we will utilize the above lemma, is that we will start out with a category in which we are interested, which will usually have a geometric flavor, and find that it does not have enough objects or does not have all of the things we want. We will extend to the category of presheaves on our original category which properly contains our original category but which loses no information since the Hom-sets are the same. The category of presheaves contains many more objects and desired properties.

2.2 Limits

Definition 2.7. Let $\mathscr{C}$ be a category. Given a diagram in $\mathscr{C}$, which consists of objects $\{D_i\}_{i\in I}$ and morphisms $\{f_\alpha\}$ where $f_\alpha : D_{d(\alpha)} \to D_{r(\alpha)}$, we say that $E = \varinjlim D_i$, the *direct limit* or *inductive limit* or *colimit*, if we have morphisms

$$\varphi_i : D_i \to E$$

such that for all f_α

$$\begin{tikzcd}
D_{d(\alpha)} \arrow[rr, "\varphi_{d(\alpha)}"] \arrow[dr, "f(\alpha)"'] & & E \\
& D_{r(\alpha)} \arrow[ur, "\varphi_{r(\alpha)}"'] &
\end{tikzcd}$$

commutes, and such that if F is any object with maps $\psi_i : D_i \to F$ such that for all f_α

$$\begin{tikzcd}
D_{d(\alpha)} \arrow[rr, "\psi_{d(\alpha)}"] \arrow[dr, "f(\alpha)"'] & & F \\
& D_{r(\alpha)} \arrow[ur, "\psi_{r(\alpha)}"'] &
\end{tikzcd}$$

commutes then there exists a unique morphism $E \to F$ such that

$$\begin{tikzcd}
& & F \\
D_{d(\alpha)} \arrow[urr] \arrow[r] \arrow[dr, "f(\alpha)"'] & E \arrow[ur, dotted] & \\
& D_{r(\alpha)} \arrow[u] \arrow[uur] &
\end{tikzcd}$$

commutes. Similarly, we may define the *inverse limit* or *projective limit* or *limit*, written $\varprojlim D_i$, by using the same definition as above, reversing the direction of the φ_i and ψ_i.

2.2.1 Examples

• Let $\mathscr{C} = \textbf{Sets}$. Let $\{S_1, S_2\}$ be our collection of objects and let $\emptyset$ be our collection of morphisms. Then $S_1 \sqcup S_2 = \varinjlim S_i$ and $S_1 \times S_2 = \varprojlim S_i$.

• Let $\mathscr{C} = \textbf{Sets}$. Let

$$\begin{tikzcd}
& S_1 \\
S_0 \arrow[ur] \arrow[dr] & \\
& S_2
\end{tikzcd}$$

be our collection of objects and morphisms. Then $\varinjlim S_i = S_1 \sqcup_{S_0} S_2$ and $\varprojlim S_i = S_1 \times_{S_0} S_2$.

3 Schemes and Topology

As we have seen, for a topological space X, we may consider the category $\mathbf{Op}(X)$ of open sets of X. We have

$$\mathcal{P}re(\mathbf{Op}(X)) = \mathbf{Fun}(\mathbf{Op}(X)^{\mathrm{op}}, \mathbf{Sets}).$$

Example 3.1. Let $F : \mathbf{Op}(X)^{\mathrm{op}} \to \mathbf{Sets}$ defined by $U \mapsto F(U) = \mathrm{Cont}(U, \mathbb{R})$, all continuous real-valued functions on U. For $U = U_1 \cup U_2$, we have

$$F(U) = \{f_1 \in F(U_1), f_2 \in F(U_2) \mid f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}\}.$$

In general we may consider arbitrary covers $\{U_i \to U\}_{i \in I}$.

Definition 3.2. A presheaf F on $\mathbf{Op}(X)$ is called a *sheaf* if for all open covers $\{U_i \to U\}_{i \in I}$, $F(U) \cong \{(f_i) \in \prod F(U_i) \mid f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}\}$.

We will use the term "a sheaf on X" to mean "a sheaf on $\mathbf{Op}(X)$". Alternately, F is a sheaf if we have

$$F(U) = \varprojlim\Big(\prod_i F(U_i) \rightrightarrows \prod_{(i,j)} F(U_i \cap U_j)\Big).$$

Notice that for any sheaf F, we have

$$F(\emptyset) = \varprojlim\Big(\prod_{\emptyset} \to \prod_{\emptyset}\Big) = *.$$

3.0.2 The Category Top

Each object in **Top** has a notion of an open cover.

Definition 3.3. Let $U \in \mathrm{Ob}(\mathbf{Top})$. An open cover $\{U_i \to U\}$ is a collection of open subsets $U_i \subseteq U$ together with inclusion maps.

Definition 3.4. A presheaf F on **Top** is a sheaf if we have the same definition given above, for all U and for all open covers.

3.0.3 Examples

- $F(U) = \mathrm{Cont}(U, \mathbb{R})$ or $F(U) = \mathrm{Cont}(U, X)$ for $X \in \mathrm{Ob}(\mathbf{Top})$. Observe that for any $X \in \mathrm{Ob}(\mathbf{Top})$, the presheaf F_X associated to X is actually a sheaf.

- Let $F(U) = H^i(U, \mathbb{R})$. Then F defines a presheaf but not a sheaf.

We now come to the notion of a scheme. The category **Schemes** is an algebraic analogue of the category of manifolds. In the case of real manifolds, an arbitrary manifold is obtained by gluing open subsets of $\mathbb{R}^n$. There is an analogous construction of general schemes from the gluing of affine schemes. As we have seen, we have an equivalence $\mathbf{AffSch} = \mathbf{CommRings}^{\mathrm{op}}$ given by $R \mapsto \operatorname{Spec} R$. Affine schemes are the spaces whose geometry is encoded in the "good/regular" functions on the space. For example, a ring R is the ring of regular functions on the affine scheme $\operatorname{Spec} R$. This is similar to the case of $\mathbb{R}^n$, in that if one knows all of smooth or continuous functions on $\mathbb{R}^n$ one, in particular, has the coordinate functions and can thus reconstruct $\mathbb{R}^n$.

Example 3.5. Let $R = \mathbb{C}[x_1, ..., x_n]$. Then $\operatorname{Spec} R = \mathbb{A}^n_{\mathbb{C}}$, affine n-space. Any geometry that we would like to study on $\mathbb{A}^n_{\mathbb{C}}$ can be studied in the polynomial ring R.

Basic open subsets of $\operatorname{Spec} R$ are given by inclusion maps

$$\operatorname{Spec} R_f \to \operatorname{Spec} R$$

corresponding to the ring maps

$$R \to R_f = R[f^{-1}] = R[x]/(xf - 1).$$

Often the notation $\operatorname{Spec} R_f = D_f$ is used. Notice that $\{\operatorname{Spec} R_{f_i} \to \operatorname{Spec} R\}$ is a cover if $(f_i) = R$. Indeed, $\bigcup \operatorname{Spec} R_{f_i}$ is the open set on which at least one of the f_i's is nonzero. The complement is the set where all f_i's vanish. Since (f_i) generates R, we may write $1 = \sum a_i f_i$. If we have a point at which all f_i's evaluate to zero, then we have $1 = 0$, so there cannot be such a point.

We have now given the collection of affine schemes a notion of a topology, what it means to have an open inclusion and ultimately what it means to have a cover. With this notion of an open cover we now have a notion of a sheaf on **AffSch**. In the case of manifolds, we would like to glue open subsets of $\mathbb{R}^n$ to obtain our manifold, but our manifold will not necessarily be an open set in $\mathbb{R}^n$. We must look outside of our original category to get that manifold. To say a manifold X is the result of gluing open sets in $\mathbb{R}^n$, means that $X = \varinjlim U$. In the category of open subsets of $\mathbb{R}^n$, one does not necessarily have all of these limits. If we move to the category of presheaves, taking limits does not create such a problem. Unfortunately, they are not necessarily the desired limits. The limits taken in the category of presheaves may not agree with the limits that exist in the original category. The way around this problem is look at the category of sheaves, as opposed to presheaves. We are going to have limits of sheaves but the limits will be determined by local data. If X is a manifold that is a limit in the category of presheaves of U_0, U_1, U_2 then

$$\operatorname{Hom}(Y, X) = \operatorname{Hom}(Y, U_1) \sqcup_{\operatorname{Hom}(Y, U_0)} \operatorname{Hom}(Y, U_2).$$

However, this is not the right way of gluing the Hom-sets. One must define maps into X locally on the domain space.

How do we construct the manifold X? We consider $\mathcal{S}hv(\mathbf{Op}(\mathbb{R}^n))$. Corresponding to each of the objects U_0, U_1, U_2, we have the presheaves F_{U_0}, F_{U_1}, and F_{U_2} which they represent. These presheaves happen to be sheaves as we have seen previously. Define $X = \varinjlim\, (F_{U_0} \rightrightarrows F_{U_1} \sqcup F_{U_2})$ as a limit in $\mathcal{S}hv(\mathbf{Op}(\mathbb{R}^n))$ as opposed to presheaves.

Definition 3.6. The category **Schemes** is the subset of $\mathcal{S}hv(\mathbf{AffSch})$ generated by $\mathbf{AffSch} \hookrightarrow \mathcal{S}hv(\mathbf{AffSch})$ via $X \mapsto F_X$ and limits.

We would like to do topology with objects like $\mathbb{A}^n_{\mathbb{C}}$. However, it is much more difficult to get our hands on the topology from the algebra. The way that we will try to repair this gap is the following: A scheme is a certain kind of presheaf **AffSch** $\to$ **Sets**. Let $\operatorname{Spec}\mathbb{C}[x,y] = F$. Then $F(\operatorname{Spec}\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}[x,y]) = \operatorname{Hom}(\mathbb{C}[x,y],\mathbb{C})$. Now, a map in $\operatorname{Hom}(\mathbb{C}[x,y],\mathbb{C})$ is just an evaluation map $x \mapsto a$ and $y \mapsto b$, which gives a pair (a,b). The problem with points is that they are discrete. This does not give a topology, only a set. We would like to give a presheaf **AffSch** $\to$ **Top** instead. But it is difficult to algebratize **Top** in a natural way. Instead, we will deal with the category of simplicial sets which we will denote **sSets**. Simplicial sets are a combinatorial way to manufacture topological spaces by abstractly gluing simplices together. The advantage is that in some way **Top** and **sSets** are equivalent, and it is much easier to give a presheaf **AffSch** $\to$ **sSets**. This will lead to the study of simplicial schemes, which will be our main objects underlying the necessary machinery.

4 Grothendieck Topology and Simplicial Sheaves

Recall that a presheaf F is a sheaf if given any open cover $\{U_i \to U\}_{i\in I}$ we have an equilizer diagram

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod F(U_i \cap U_j).$$

Definition 4.1. Let $\mathscr{C}$ be a category. A *Grothendieck topology* T on $\mathscr{C}$ is a collection $\mathrm{cov}(T)$ of sets of morphisms $\{U_i \to U\}_{i\in I}$ in $\mathscr{C}$ which are called *coverings*, satisfying the following:
(T1) If $\{U_i \to U\} \in \mathrm{cov}(T)$ and $V \to U$ is any morphism then $U_i \times_U V$ exists for each i and $\{U_i \times_U V \to V\}$ is a covering.
(T2) If $\{U_i \to U\}$ and $\{V_{ij} \to U_i\}$ are coverings then $\{V_{ij} \to U\}$ is a covering.
(T3) If $\varphi : U \to U'$ is an isomorphism then $\{U \to U'\}$ is a covering.

Definition 4.2. A *site* is a pair $(\mathscr{C}, T)$ consisting of a category $\mathscr{C}$ and a Grothendieck topology T on $\mathscr{C}$.

4.0.4 Examples

- $\mathbf{Op}(X)$.

- **Sets** with surjective families of maps.

- **Schemes**.

- G-**Sets**, the category of sets admitting a G-action for a group G.

Definition 4.3. A *sheaf* on a site $(\mathscr{C}, T)$ is a presheaf $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ such that if $\{U_i \to U\} \in \mathrm{cov}(T)$, then

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j)$$

is an equalizer diagram for all covers.